Practice Questions for Final Exam

Economic Growth
Spring 2005

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Second Exam from Fall 2004

Question I. (25 points) Consider the model of human capital. Output is produced according to the production function $Y = K^{\alpha}(huL)^{1-\alpha}$, where $0 < \alpha < 1$ and where u is the fraction of time that each person spends working. A constant fraction s of output is invested in new physical capital, so that physical capital accumulation is given by

$$\dot{K}(t) = sK(t)^{\alpha}[h(t)uL(t)]^{1-\alpha} - \delta K(t).$$

The labor force $L(t)$ grows at the constant rate $n > 0$. Human capital accumulation is given by

$$\dot{h}(t) = B(1-u)h(t).$$

a) (5 points) Derive the differential equations for $k = \frac{K}{L}$ and $\widehat{k} = \frac{k}{h}$.

b) (10 points) Do the following comparative dynamics exercise: $\delta' < \delta$. In other words, in the modified economy the depreciation rate for physical capital is lower than in the baseline economy.

Draw the Solow-like diagram for the variable $\widehat{k}$ and the phase diagram for (k, h). On each diagram, indicate clearly what changes in the modified case.

c) (10 points) Draw the time paths of $\ln(k)$, $\ln(h)$ and of $\ln(y)$ for both the baseline and the modified cases. Be sure to label the slopes of all lines in your graphs.

Question II. (55 points) Consider the Ak model. As in class, the production function of an individual firm depends on the total amount of capital in the economy $\overline{K}$, and is given by

$$Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}\overline{K}(t)^{1-\alpha}.$$

The utility function has the usual form (constant intertemporal elasticity of substitution)

$$u[c(t)] = \frac{c(t)^{1-\theta} - 1}{1-\theta}.$$

Assume there is no population growth $(n = 0)$ and normalize the size of the population to $N = 1$.

Suppose the government decides to subsidize the banking sector. For each unit of assets that a bank holds at time t, the government pays the bank an amount ϕ. Assume that the government taxes the labor income of households at rate $\tau(t)$ in order to balance its budget.

a) (5 points) Write down the household's complete optimization problem. Be sure to include all of the constraints.

b) (5 points) Write the differential equations for c and a that solve this problem.

[You do not need to show your derivations if you know the answer. If you are not sure of the answer, use the usual Hamiltonian approach.]

c) (5 points) Write the profit-maximization problem of the representative firm and the first-order conditions for this problem. Convert these equations to intensive form.

d) (5 points) What are the equilibrium conditions for this economy?

e) (5 points) What is the government's budget constraint?

f) (5 points) Using your answers above, derive the differential equations for c and k that characterize the equilibrium of this economy.

g) (10 points) Do the following comparative dynamics exercise: In the baseline economy there is no subsidy $(\phi = 0)$. In the modified economy, there is a positive subsidy $(\phi > 0)$. Both economies start at time $t = 0$ with the same amount of capital k_0.

Draw the phase diagram, including both the baseline and the modified economy on the same diagram. Be sure to label your diagram clearly.

h) (10 points) Draw the time paths of k and of c for both the baseline and the modified economy. Be sure to label your diagrams clearly, including the slopes of all curves.

i) (5 points) Is the initial level of consumption $c(0)$ higher in the baseline case or the modified case? Why? [Give a short, **intuitive** explanation.]

Question III. (20 points) Consider the Romer model, where output is produced using different types of capital according to the production function

$$Y(t) = AL(t)^{1-\alpha} \int_0^{M(t)} K_j(t)^{\alpha} \, dj.$$

Suppose that the government taxes the production of the final output good. In particular, for each unit of output that the representative final-output producer makes, it must pay the government an amount ϕ.

a) (5 points) Write the optimization problem of the final output producer, and use this problem to derive the demand function for capital of type j.

b) (5 points) Write the optimization problem of a capital producer (firm j), and use this problem to derive the optimal quantity K_j^* for firm j to sell and the optimal price R_j^* for firm j to charge.

c) (5 points) What are the equilibrium conditions for this economy?

d) (5 points) Is the equilibrium interest rate r in this model (a) increasing in ϕ, or (b) decreasing in ϕ, or (c) unaffected by ϕ? Why? [Answer by choosing (a) , (b) , or (c) and giving a short, **intuitive** explanation.]

Second Exam from Spring 2004

Part I. (60 points) Consider the Ak model, where there are externalities in production. As in class, the production function of each firm depends on the total amount of capital in the economy $\overline{K}$, and is given by

$$Y(t) = AK(t)^{\alpha} L(t)^{1-\alpha} \overline{K}(t)^{1-\alpha}.$$

Assume there is no population growth $(n = 0)$ and normalize the size of the population to $N = 1$. The utility function has the usual form. Suppose this economy receives **foreign aid**: At every point in time t the representative household receives a payment equal to $\phi\overline{K}(t)$ from the World Bank. In other words, the size of the payment depends on the total amount of capital in the economy. (This implies that the aid payment will grow as the economy grows.) In this question you will analyze the effect of an aid payment in the Ak model.

a) Write down the household's complete optimization problem. Be sure to include all constraints.

b) Write the pair of differential equations for c and a that solve this problem.

[You do not need to show your derivations if you know the answer. If you are not sure of the answer, use the usual Hamiltonian approach.]

c) Write down the profit-maximization problem of a typical firm and the first-order conditions for this problem. Convert these equations to intensive form.

d) What are the equilibrium conditions for this economy?

e) Using your answers above, derive the differential equations for c and k that characterize the equilibrium of this economy.

f) Do the following comparative dynamics exercise: The baseline economy receives no aid $(\phi = 0)$, and the modified economy receives a positive amount of aid $(\phi > 0)$. Both economies start at time $t = 0$ with the same amount of capital k_0.

Draw the phase diagram, including both the baseline and the modified economy on the same diagram. Be sure to label your diagram clearly. [Also draw any other diagrams that are useful.]

g) Draw the time paths of k and of c for both the baseline and the modified economy. Be sure to label your diagrams clearly.

h) Can the amount of foreign aid ϕ be chosen so that the equilibrium of the economy with aid is optimal? Why or why not? (Give a **short** intuitive explanation; you do not need to solve an optimal growth problem.)

Part II. (30 points) Consider the model of human capital. Output is produced according to the production function $Y = K^{\alpha} (huL)^{1-\alpha}$, where $0 < \alpha < 1$ and where u is the fraction of time that each person spends working. A constant fraction s of output is invested in new physical capital, so that physical capital accumulation is given by

$$\dot{K}(t) = sK(t)^{\alpha} [h(t) uL(t)]^{1-\alpha} - \delta K(t) .$$

Human capital accumulation is given by $\dot{h}(t) = B(1-u) h(t)$. The labor force $L(t)$ grows at the constant rate $n > 0$.

a) Derive the differential equations for $k = \frac{K}{L}$ and $\widehat{k} = \frac{k}{h}$.

b) Do the following comparative dynamics exercise: $n' > n$. As usual, the baseline economy is on the balanced growth path at $t = 0$. The modified economy starts at $t = 0$ with the same amounts of physical capital and human capital as the baseline economy.

Draw the phase diagram for (k, h) and the Solow-like diagram for $\widehat{k}$, including both the baseline and the modified economies on the same diagrams. Be sure to label all of the lines and curves in your diagrams clearly.

c) Draw the time paths of h, k, and y for both the baseline and the modified economy.

Part III. (10 points) Consider the Romer model, where output is produced using different types of capital according to the production function

$$Y(t) = AL(t)^{1-\alpha} \int_0^{M(t)} K_j(t)^{\alpha} \, dj.$$

Suppose that the government imposes a **tax on profits**: All firms (both final output producers and capital producers) must pay a fraction τ of their profits to the government.

In this question, we are only going to look at the behavior of firms.

a) Write the optimization problem of the representative final output producer, and use this problem to derive the demand function for capital of type j.

b) Write the optimization problem of a capital producer (firm j), and use this problem to derive the optimal quantity K_j^* for firm j to sell and the optimal price R_j^* for firm j to charge.