BANKING AND FINANCIAL FRAGILITY

A Baseline Model: Diamond and Dybvig (1983)

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Objective

- Want to develop a model to help us understand:
  - why banks and other financial institutions tend to have a maturity mismatch between their assets and liabilities
  - in what way(s) this maturity mismatch can create the type of financial crises we see in reality

- ...and use this model to evaluate policy proposals

- Our model will be very simple in some dimensions
  - but we will get a remarkable amount of mileage out of it

Readings:
- Diamond & Dybvig (JPE, 1983)
- Allen & Gale, chapter 3
Outline

1. The Environment
2. Autarky
3. The Efficient Allocation
4. Banking
5. Two Views of Financial Fragility
6. Summary
1. The Environment
1.1 Time and commodities

- 3 time periods
  - $t = 0, 1, 2$

- Single consumption good in each period
1.2 Economic agents

- Continuum of investors, \( i \in [0,1] \)
- Each is endowed with 1 unit of the good at \( t = 0 \)
  - and nothing at \( t = 1,2 \)
- Each has utility function
  \[
  \begin{cases}
    u(c_1^i) & \text{if investor } i \text{ is type 1 \text{ - "impatient"}} \\
    u(c_2^i) & \text{type 2 \text{ - "patient"}}
  \end{cases}
  \]
- Denote type by \( \omega_i \in \Omega = \{1,2\} \)
- At \( t = 0 \), investor does not know her type
  - learns type at \( t = 1 \)
  - type is private information
Uncertainty

- Each investor will be impatient with probability $\lambda \in (0,1)$
- $\lambda$ also = fraction of all investors who will be impatient
  - no aggregate uncertainty here
  - only uncertainty is about *which* investors will be impatient

Consumption plans

- A consumption plan for investor $i$ is
  $$c^i = (c_1^i, c_2^i) \in \mathbb{R}_+^2$$
1.3 Technologies

- Two assets for transforming $t = 0$ goods to later periods
  - Storage:
    
    \[ \begin{align*}
    & \text{1 unit at} \quad \begin{cases} t = 0 \\ t = 1 \end{cases} \\
    \quad \text{yields} \quad \begin{cases} 1 \text{ at } t = 1 \\ 1 \text{ at } t = 2 \end{cases} \end{align*} \]

  - Investment:
    
    \[ \begin{align*}
    & \text{1 unit at } t = 0 \quad \text{yields} \quad \begin{cases} r < 1 \text{ at } t = 1 \\ R > 1 \text{ at } t = 2 \end{cases} \end{align*} \]

- investment can only be started at $t = 0$
- $(1 - r) = \text{“liquidation cost”}$
2. Allocations under Autarky
Suppose there is no trade

- each investor divides her endowment at \( t = 0 \) between storage and investment
- consumes the proceeds at either \( t = 1 \) or \( t = 2 \)

Let \( x = \) amount placed into investment

- \((1 - x)\) is placed into storage

Investor’s objective: \( \max_{\{x\}} \lambda u(c_1) + (1 - \lambda)u(c_2) \)

Feasibility constraints:

\[
c_1 = rx + (1 - x) = 1 - (1 - r)x
\]

\[
c_2 = Rx + (1 - x) = 1 + (R - 1)x
\]
Restating the investor’s maximization problem:

\[
\max_{\{x \in [0,1]\}} \lambda u(c_1) + (1 - \lambda) u(c_2)
\]

subject to

\[
c_1 = 1 - (1 - r)x
\]

\[
c_2 = 1 + (R - 1)x
\]

Q: Is this allocation Pareto optimal?
3. The (full information) efficient allocation
3.1 Definitions

- An **allocation** is a list of consumption plans:
  \[ \{(c_1^i, c_2^i)\}_{i \in [0,1]} \]

- An allocation is **symmetric** if
  \[ (c_1^i, c_2^i) = (c_1^j, c_2^j) \] for all \( i, j \)
  - characterized by only two numbers

- Under **full information**, investors’ preference types are observable (to the planner)

**Q:** What is the best symmetric allocation the planner can implement under full information?
3.2 Some properties of efficient allocations

- The efficient allocation of resources in this environment requires:
  - no investment should be liquidated at \( t = 1 \)
  - no storage should be held until \( t = 2 \)
    - recall that there is no aggregate uncertainty here

- In our notation:
  \[
  \lambda c_1 = 1 - x \\
  (1 - \lambda)c_2 = Rx
  \]

- Combining to eliminate \( x \):
  \[
  \lambda c_1 + (1 - \lambda) \frac{c_2}{R} = 1
  \]
Repeating

\[ \lambda c_1 + (1 - \lambda) \frac{c_2}{R} = 1 \]

⇒ The planner can do better than autarky (Why?)
3.3 Finding the best symmetric allocation

The full-information efficient allocation solves

$$\max \limits_{\{c_1, c_2\}} \lambda u(c_1) + (1 - \lambda)u(c_2)$$

subject to

$$\lambda c_1 + (1 - \lambda) \frac{c_2}{R} = 1$$

multiplier = $\mu$

First-order conditions:

$$\lambda u'(c_1) = \lambda \mu$$

$$(1 - \lambda)u'(c_2) = (1 - \lambda) \frac{\mu}{R}$$

or

$$u'(c_1) = Ru'(c_2)$$

Solution:

$$(c_1^*, c_2^*) \text{ with } c_1^* < c_2^*$$
Depending on the function \( u \), we can have

\[ x^* = (1 - \lambda) \frac{c_2^*}{R} \]

or

\[ (1 - x^*) = \lambda c_1^* \]
Exercises

- We know \((c_1^*, c_2^*)\) solves:

  \[
  \max_{\{c_1, c_2\}} \lambda u(c_1) + (1 - \lambda) u(c_2)
  \]

  subject to \(\lambda c_1 + (1 - \lambda) \frac{c_2}{R} = 1\)

- Find \((c_1^*, c_2^*)\) for the following utility functions:

  - \(u(c) = \ln(c)\)  \hspace{1cm} A: \((c_1^*, c_2^*) = (1, R)\)
  - \(u(c) = c\)  \hspace{1cm} (risk neutral)  \hspace{1cm} A: \((c_1^*, c_2^*) = (0, \frac{R}{1-\lambda})\)
4. Banking
4.1 More on the environment

- Return to the case where types are private information
- Investors can meet at $t = 0$, but are isolated from each other at $t = 1$
  - cannot trade with each other
- Each investor can visit a central location at $t = 1$ before consuming
  - arrive one at a time
  - must consume when they arrive (ice cream on a hot day)
- These assumptions aim to capture transaction needs
  - when a consumption opportunity arises, investors cannot quickly sell illiquid assets
4.2 A banking arrangement

- Suppose a bank opens at $t = 0$, offers the following deal:
  - deposit at $t = 0 \Rightarrow$ you can withdraw at either $t = 1$ or $t = 2$ (your choice)

- Bank places a fraction $x^*$ of its assets into investment

- Investors who choose $t = 1$ will receive $c_1^*$
  - as long as the bank has funds available

- Investors who choose $t = 2$ will receive an even share of the bank’s matured assets

- These rules create a withdrawal game
  - each investor decides when to withdraw
  - payoffs depend on the choices made by all investors
4.3 Withdrawal strategies

First: impatient investors will always withdraw at $t = 1$
  - do not value consumption at $t = 2$

$\Rightarrow$ We only need to determine what an investor will do in the event she is patient

A withdrawal strategy is:

$$y_i \in \{1, 2\}$$

where $y_i = t$ means withdraw in period $t$ when patient

More notation:

- $y = \{y_i\}_{i \in [0, 1]}$ is a complete profile of withdrawal strategies
- $y_{-i} =$ profile of strategies for all investors except $i$
4.4 Best responses

- Suppose an investor anticipates $y_{-i} = 2$
  - that is, all other investors will withdraw at $t = 2$ when patient

- What is her best response?
  - if she withdraws at $t = 1$: $c_1^*$
  - if she withdraws at $t = 2$: even share of matured investment
  - what is this even share worth?
    
    \[
    \frac{Rx^*}{1 - \lambda} = \frac{(1 - \lambda)c_2^*}{1 - \lambda} = c_2^*
    \]

- We know $c_2^* > c_1^* \implies$ best response $y_i = 2$
4.5 Equilibrium

A Nash equilibrium is a profile of withdrawal strategies \( y^* \) such that, for all \( i \), \( y_i^* \) is a best response to \( y_{-i}^* \).

- focus on symmetric equilibria in pure strategies

Result 1: There is a Nash equilibrium with

\[ y_i = 2 \quad \text{for all } i. \]

- In this equilibrium:
  - impatient investors withdraw at \( t = 1 \), receive \( c_1^* \)
  - patient investors withdraw at \( t = 2 \), receive \( c_2^* \)
  \( \Rightarrow \) implements the (full information) efficient allocation
  - even though types are private information (!!)
4.6 Interpretations

- Notice what the bank is doing in this model
  - issuing demand deposits
  - while holding (some) illiquid assets

- Why is this activity socially desirable?
  - because investors face uncertainty about their liquidity needs
  - bank allows all investors to hold liquid claims

- This activity is often called “maturity transformation”
  - emphasize that this a productive activity
    - bank is “producing” liquidity
  - also called “fractional reserve banking”
Suppose we construct the balance sheet of this bank

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment</td>
<td>Deposits</td>
</tr>
<tr>
<td>$R x^*$</td>
<td>$c_1^*$</td>
</tr>
<tr>
<td>Storage</td>
<td>Equity</td>
</tr>
<tr>
<td>$1 - x^*$</td>
<td>$E$</td>
</tr>
</tbody>
</table>

note that investment is valued at “hold to maturity” price

Equity (or “bank capital”) is defined as Assets – Liabilities

$$E \equiv R x^* + (1 - x^*) - c_1^*$$

A bank is said to be solvent if $E \geq 0$

- by design, our banking arrangement is solvent
- even though some of the bank’s assets are illiquid
5. Two views of financial fragility
So far: it can be socially useful to have banks doing maturity transformation

- allows all investors to hold liquid claims
- while (partially) benefitting from the higher return on illiquid investment

In practice, maturity transformation appears to be at the center of many financial crises

What does our model say about the fragility of this banking arrangement?

We can see two views of what happens during a crisis
5.1 Self-fulfilling bank runs

Q: Does the withdrawal game have other equilibria?

- Suppose investor $i$ anticipates:
  \[ y_{-i} = 1 \]
  - everyone else will “run” and withdraw at first opportunity

- What is her best response?
  - the bank will start liquidating investment …
  - should she join the run?

More generally:

- Find the best response of investor $i$ to any profile $y_{-i}$
For any \( y_{-i} \), define:

\[
e(y_{-i}) = \text{number of } t = 1 \text{ withdrawals that will be made by patient investors ("extra" withdrawals at } t = 1)\]

- equals number of investors who have \( y_i = 1 \) and are patient
- note: \( e \in [0,1 - \lambda] \)

To find best response of investor \( i \):

- compare expected payoffs of withdrawing at \( t = 1 \) and \( t = 2 \)
- both of these payoffs will depend on \( e \)
If a patient investor chooses $t = 1$, she receives $c_1^*$ ...

... if (and only if) bank has funds available when she arrives

If she chooses $t = 2$, she receives:

- an even share of the bank’s remaining (matured) assets
- critical question: what is this even share worth?

At $t = 2$, the bank will have:

$$1 - x^* - \lambda c_1^* + R \left( x^* - e \frac{c_1^*}{r} \right)$$

- $1 - x^* - \lambda c_1^*$: storage and first $\lambda$ withdrawals
- $R \left( x^* - e \frac{c_1^*}{r} \right)$: investment liquidated for extra $t = 1$ withdrawals

= 0
Repeating: the bank will have

\[ R \left( x^* - e \frac{c_1^*}{r} \right) \]

Number of remaining investors: \( 1 - \lambda - e \)

An even share is worth:

\[ c_2(e) = \max \left\{ \frac{R \left( x^* - e \frac{c_1^*}{r} \right)}{1 - \lambda - e}, 0 \right\} \]

Q: What does this function look like?

Note:

\[ c_2(0) = \frac{Rx^*}{1 - \lambda} = c_2^* \] (as before)
Assume

\[ c_1^* > 1 - (1 - r)x^* \]  \hspace{1cm} (A1)

- this condition implies the bank is “illiquid”
  - it cannot afford to give \( c_1^* \) to all investors at \( t = 1 \)

Then (you can verify):

\[ \frac{dc_2(e)}{de} < 0 \]

and

\[ c_2(e) = 0 \text{ for some } e < 1 - \lambda \]

and

\[ c_2(e) \text{ is strictly concave on } (0, e^B) \]
Graphically:

Define: $e^T$ ("threshold") so that \[ c_2(e^T) = c_1^* \]

Define: $e^B$ ("bankruptcy") so that \[ c_2(e^B) = 0 \]
Summarizing investor $i$’s payoffs:

For any $y_{-i}$, the best response of investor $i$ is:

- If $e(y_{-i}) \leq e^T$, then $y_i = \begin{cases} 2 \\ 1 \end{cases}$
- If $e(y_{-i}) \geq e^T$, then $y_i = \begin{cases} 1 \\ 2 \end{cases}$

Result 2: There is also a Nash equilibrium with $y_i = 1$ for all $i$. 
This second equilibrium resembles the bank runs we have seen during financial crises

- a “panic”, but with fully rational investors
- nothing fundamental is wrong; bank is still solvent
- the crisis is (simply) a result of self-fulfilling beliefs

Another look at the balance sheet:

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If assets are valued at liquidation prices, equity becomes

$$\hat{E} \equiv rx^* + (1 - x^*) - c_1^* < 0$$
A bank is **solvent** if \( E \geq 0 \); otherwise it is **insolvent** (repeat)

A bank is **liquid** if \( \hat{E} \geq 0 \); otherwise it is **illiquid** (new)

**Results 1 and 2**: When a bank is solvent but illiquid, the withdrawal game has (at least) two equilibria:

- \( y_i = 2 \) for all \( i \): implements the planner’s allocation \((c_1^*, c_2^*)\)
- \( y_i = 1 \) for all \( i \): a bank run

“self-fulfilling financial fragility”
Properties of the bank-run equilibrium:

- Fraction of investors served:

  \[ q \equiv \frac{\text{total assets}}{\text{amount per investor}} = \frac{1 - (1 - r)x^*}{c_1^*} < 1 \]

- Expected utility in the bank-run equilibrium:

  \[ qu(c_1^*) + (1 - q)u(0) < u(qc_1^* + (1 - q)0) \]
  \[ = u(1 - (1 - r)x^*) \]
  \[ < u(1) \]
  \[ \leq u(\text{autarky}) \quad (!) \]

- Outcome is worse than having no bank at all
5.2 Bad news and bank runs

- Suppose at $t = 1$ investors learn the return on investment has fallen to $R_L < R$
  - unexpected shock (for simplicity)
  - banking contract (that is, $x^*, c_1^*$) is already fixed

- An investor who withdraws at $t = 2$ now receives

$$c_2(e) = \max \left\{ R_L \left( x^* - e \frac{c_1^*}{r} \right), 0 \right\}$$

- Focus on:

$$c_2(0) = \frac{R_L x^*}{1 - \lambda}$$
Consider two possibilities:

- $R_{L'} < R_L < R$

At $R_L$, there are two equilibria, as before.

At $R_{L'}$, withdrawing at $t = 1$ is a dominant strategy!

⇒ A bank run is the unique Nash equilibrium.
How low must $R_L$ be for withdrawing at $t = 1$ to become a dominant strategy?

Start with

$$c_2(0) = \frac{R_L x^*}{1-\lambda}$$

Using

$$x^* = (1 - \lambda) \frac{c_2^*}{R},$$

we have

$$c_2(0) = \frac{R_L}{R} c_2^*$$

Withdrawing at $t = 1$ is a dominant strategy if:

$$c_2(0) < c_1^*$$

or

$$R_L < \frac{c_1^*}{c_2^*} R \equiv \overline{R}_L$$
Another view

- “hold to maturity” value of investment has fallen
- equity is now:
  \[ E = R_L x^* + (1 - x^*) - c_1^* \]
- (Verify:) \( R_L < \bar{R}_L \iff E < 0 \)
  - if the loss is large enough to make the bank insolvent …
  - … withdrawing at \( t = 1 \) is a dominant strategy

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**Result 3:** If $R_L < \bar{R}_L$, the *unique* Nash equilibrium strategy profile is

$$y_i = 1 \text{ for all } i.$$ 

- If the bank is insolvent, arrangement *necessarily* collapses
  - if $c_1^*$ is close to $c_2^*$, the required losses would be very small

- Fraction of investors served in the run:

  $$q = \frac{1-(1-r)x^*}{c_1^*} \text{ independent of } R_L!$$

  - Why? Because during a run, all investment is liquidated
  - same as when the run was based on self-fulfilling beliefs
An example:

- \( u(c) = \ln(c) \) \quad \Rightarrow \quad \text{verify: } (c_1^*, c_2^*) = (1, R)
- \text{also: } r = \frac{1}{2}, \quad \lambda = \frac{1}{2} \quad \Rightarrow \quad \text{verify: } x^* = \frac{1}{2}
- \text{then (verify) } \overline{R_L} = 1

Suppose \( R_L = 0.99 \)

- it is socially feasible to give all investors (almost) 1 unit

The equilibrium allocation gives 1 to a fraction

\[
q = \frac{1 - (1 - r)x^*}{c_1^*} = \frac{3}{4}
\]

- and nothing to the remaining 1/4 \quad \text{(much worse!)}
6. Summary
Takeaways from Diamond & Dybvig (1983)

- Maturity transformation is socially useful ...
  - D&D gave us a good model for thinking about where the value comes from
  - banks are in the business of “creating” liquidity

- ... but makes banks fragile

- Two ways of thinking about this fragility
  - a bank that is solvent but illiquid is *susceptible* to a run
    - a loss of confidence – for whatever reason – leads to a run
  - a bank that is insolvent will *necessarily* have a run
    - small losses on a bank’s assets can have large consequences
References and further reading


- see especially Chapters 3 and 5
