Supplemental Appendix for:
"Allocating Losses: Bail-ins, Bailouts and Bank Regulation"

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This appendix provides proofs of the propositions presented in the paper.

Proposition 1. ***(Planner's allocation)*** *The efficient plan* $\left(h^*, \hat{h}^*, b^*\right)$ *sets*

$$h^* = \hat{h}^* = \left\{ \begin{array}{c} \lambda \\ \lambda^* \end{array} \right\} \quad \textit{and} \quad b^* = \left\{ \begin{array}{c} 0 \\ \lambda - \lambda^* \end{array} \right\} \quad \textit{as} \quad \lambda \left\{ \begin{array}{c} \leq \\ > \end{array} \right\} \lambda^*.$$

We first establish a key property of the efficient plan in the following lemma.

Lemma 1. *The efficient plan satisfies* $h^* = \hat{h}^*$ *for all* $\lambda \in \Lambda$.

Proof. To begin, note that the resource constraint (4) will hold with equality for all λ at the solution to the planner's problem. The non-negativity restrictions then imply that the planner will set $h = \hat{h} = b = 0$ when $\lambda = 0$. When there is no loss, investors are neither bailed in nor bailed out.

For $\lambda > 0$, let θ denote the multiplier on the resource constraint (4). We can then write the first-order conditions for the optimal choice of h as[2]

$$u'\left((1-h)c_1^*\right) \geq \theta \quad \text{and} \quad h\left[u'\left((1-h)c_1^*\right) - \theta\right] = 0 \tag{20}$$

and for the optimal choice of $\hat{h}$ as

$$u'\left((1-\hat{h})c_2^*\right) \geq \frac{\theta}{R} \quad \text{and} \quad \hat{h}\left[u'\left((1-\hat{h})c_2^*\right) - \frac{\theta}{R}\right] = 0. \tag{21}$$

We will show that the solutions to these two sets of equations are necessarily the same, considering the cases of boundary and interior solutions separately.

First, suppose the solution has $h = 0$. Then equation (20) implies

$$u'\left(c_1^*\right) \geq \theta.$$

The reference allocation (c_1^*, c_2^*) is characterized by the standard optimality condition in the

[2] Note that the Inada conditions on the function u imply that the upper bounds on $h(\phi)$ and $\hat{h}(\phi)$ in equation (3) will never bind at the solution to the problem.

Diamond-Dybvig framework,

$$u'(c_1^*) = Ru'(c_2^*).$$

Combining these two equations yields

$$u'(c_2^*) \geq \frac{\theta}{R}$$

and, therefore, the unique $\hat{h}(\phi)$ satisfying the conditions in equation (21) is $\hat{h}(\phi) = 0$.

Next, suppose the solution has $h > 0$. Then equation (20) implies

$$u'((1-h)c_1^*) = \theta.$$

Given that the utility function u is of the constant-relative-risk-aversion form, the ratio of marginal utilities depends only on the ratio of consumption levels, that is, we have

$$\frac{u'(\alpha c_1^*)}{u'(\alpha c_2^*)} = \frac{u'(c_1^*)}{u'(c_2^*)} = R \tag{22}$$

for any $\alpha > 0$. These last two equations imply

$$u'((1-h)c_2^*) = \frac{\theta}{R}$$

and, therefore, setting $\hat{h} = h$ is the unique solution to equation (21). Combining these two cases, we have shown that $\hat{h} = h$ holds for all λ, which establishes the result. □

Proof of Proposition 1. Using the result from Lemma 1 and the simplified resource constraint in equation (7), we can write the planner's problem as choosing the bail-in h to maximize

$$\pi u\big((1-h)c_1^*\big) + (1-\pi)u\big((1-h)c_2^*\big) - \mu[\lambda - h],$$

where the non-negativity constraints for bail-ins and bailouts can be written as

$$0 \leq h \leq \lambda. \tag{23}$$

The objective function is strictly concave in h and has slope

$$-\big[\pi u'\big((1-h)c_1^*\big)c_1^* + (1-\pi)u'\big((1-h)c_2^*\big)c_2^*\big] + \mu$$

or

$$-u'\big((1-h)c_1^*\big)\left[\pi c_1^* + (1-\pi)\frac{u'\big((1-h)c_2^*\big)}{u'\big((1-h)c_1^*\big)}c_2^*\right] + \mu.$$

Using equation (22) and the resource constraint for the reference allocation in equation (3), it is straightforward to show the term in square brackets reduces to 1. Using equation (8) to replace μ, we can then write the slope as

$$-u'\big((1-h)c_1^*\big) + u'\big((1-\lambda^*)c_1^*\big).$$

If $\lambda \leq \lambda^*$, this slope is non-negative when evaluated at the upper bound for h in equation (23) and, therefore, the solution is $h = \lambda$. If $\lambda > \lambda^*$, the constraints in equation (23) do not bind and the solution is $h^* = \lambda^*$. In both cases, the planner's optimal bailout is determined by setting $h = h^*$ in equation (7) and solving for b^*. □

Proposition 2. ***(Bailout and remaining bail-in)*** *Given h and y, the bail-in of remaining investors $\hat{h}$ and bailout b satisfy:*

(i) if $\hat{h}_{NB}(h,y) \leq \lambda^*$, then $\hat{h}(h,y) = \hat{h}_{NB}(h,y)$ and $b(h,y) = 0$

(ii) if $\hat{h}_{NB}(h,y) > \lambda^*$, then $\hat{h}(h,y) = \lambda^*$ and and the bailout satisfies equation (14).

Proof. As a first step, we use equation (11) to write the bank's marginal value of resources after π withdrawals as

$$V_1\big(\psi(h,b),y\big) \equiv \left\{ \begin{array}{c} u'\big(R\psi(h,b)\big)R \\ \pi u'\big(\psi(h,b)c_1^*\big)c_1^* + (1-\pi)u'\big(\psi(h,b)c_2^*\big)c_2^* \end{array} \right\} \quad \text{as} \quad \left\{ \begin{array}{c} y=2 \\ y=1 \end{array} \right\}. \tag{24}$$

This expression together with the definition of ψ in equation (9) shows that V_1 is strictly decreasing in b for both values of y. In what follows, we use the expression to establish the two parts of the proposition in turn.

Part (i): First suppose $y = 2$. Then $\hat{h}_{NB}(h,2) \leq \lambda^*$ implies

$$\big(1-\hat{h}_{NB}(h,2)\big)c_2^* \geq (1-\lambda^*)\,c_2^*.$$

Using equations (9) and (13), we can rewrite the left-hand side of the inequality in terms of the bank's remaining resources ψ,

$$R\psi(h,0) \geq (1-\lambda^*)\,c_2^*,$$

which implies

$$u'\big(R\psi(h,0)\big)R \leq u'\big((1-\lambda^*)\,c_2^*\big)R.$$

Using equation (22), we can rewrite the right-hand side of this inequality as

$$u'\big((1-\lambda^*)\,c_2^*\big)R = u'\big((1-\lambda^*)\,c_1^*\big) = \mu.$$

Combining these last two equations with equation (24) yields

$$V_1\big(\psi(h,0),2\big) \leq \mu,$$

which implies $b^* = 0$ is the unique solution to the first-order condition in equation (12).

Now suppose $y = 1$. In this case, we use $\hat{h}_{NB}(h,1) \leq \lambda^*$ to write both

$$\big(1-\hat{h}_{NB}(h,1)\big)c_1^* \geq (1-\lambda^*)\,c_1^* \quad \text{and} \quad \big(1-\hat{h}_{NB}(h,1)\big)c_2^* \geq (1-\lambda^*)\,c_2^*.$$

Using equations (9) and (13), we can write the left-hand side of these inequalities in terms of the bank's remaining resources ψ,

$$\psi(h,0)c_1^* \geq (1-\lambda^*)\,c_1^* \quad \text{and} \quad \psi(h,0)c_2^* \geq (1-\lambda^*)\,c_2^*.$$

The first of these two inequalities implies

$$u'\big(\psi(h,0)c_1^*\big)c_1^* \leq u'\big((1-\lambda^*)\,c_1^*\big)c_1^* = \mu c_1^*, \tag{25}$$

while the second implies

$$\begin{aligned} u'\big(\psi(h,0)c_2^*\big)c_2^* &\leq u'\big((1-\lambda^*)\,c_2^*\big)c_2^* \\ &= u'\big((1-\lambda^*)\,c_1^*\big)\frac{c_2^*}{R} = \mu\frac{c_2^*}{R} \end{aligned} \tag{26}$$

where the first equality on the second line uses equation (22). Combining equations (25) and (26) yields

$$\pi u'\big(\psi(h,0)c_1^*\big)c_1^* + (1-\pi)u'\big(\psi(h,0)c_2^*\big)c_2^* \leq \mu\Big(\pi c_1^* + (1-\pi)\frac{c_2^*}{R}\Big) = \mu,$$

where the last equality uses the resource constraint in equation (3). Combining this inequality with equation (24) yields

$$V_1\big(\psi(h,0),1\big) \leq \mu,$$

which implies $b^* = 0$ is the unique solution to the first-order condition in equation (12) when $y = 1$ as well. When $b^* = 0$, the remaining investors will be bailed in at rate $\hat{h}_{NB}$ as defined in equation (13).

Part (*ii*): When $\hat{h}_{NB}(h,1) > \lambda^*$, the steps above show that $V_1(\psi(h,b=0),y) > \mu$ and, therefore, the solution to the fiscal authority's bailout choice problem is interior. In this case, the first-order condition in equation (12) holds with equality,

$$V_1\big(\psi(h,b),y\big) = \mu = u'\big((1-\lambda^*)\,c_1^*\big).$$

If $y = 2$, this equation can be written as

$$u'\big(R\psi(h,b)\big)R = u'\big((1-\lambda^*)\,c_1^*\big) = u'\big((1-\lambda^*)\,c_2^*\big)R,$$

where the last equality uses equation (22). Using the monotonicity of u', we have

$$R\psi(h,b) = (1-\lambda^*)\,c_2^*.$$

Using the definition of ψ in equation (9), we can rewrite this equation as

$$R\frac{1-\lambda-\pi(1-h)c_1^*+b}{1-\pi} = (1-\lambda^*)\,c_2^*.$$

Solving for b yields

$$b = (1-\pi)(1-\lambda^*)\frac{c_2^*}{R} - (1-\pi)\left(\frac{1-\lambda-\pi(1-h)c_1^*}{1-\pi}\right)$$
$$= (1-\pi)\frac{c_2^*}{R}\left(1-\lambda^* - \frac{R}{c_2^*}\,\frac{1-\lambda-\pi(1-h)c_1^*}{1-\pi}\right).$$

Finally, using equations (3) and (13), we can rewrite this equation as

$$b = (1-\pi c_1^*)\left(\hat{h}_{NB}(h,2) - \lambda^*\right),$$

as desired.

The steps for $y = 1$ are similar. Equation 6 can be written as

$$\pi u'\big(\psi(h,b)c_1^*\big)c_1^* + (1-\pi)u'\big(\psi(h,b)c_2^*\big)c_2^* = u'\big((1-\lambda^*)\,c_1^*\big).$$

Using equation (22), we can write this equation as

$$u'\big(\psi(h,b)c_1^*\big)\left(\pi c_1^* + (1-\pi)\frac{c_2^*}{R}\right) = u'\big((1-\lambda^*)\,c_1^*\big).$$

Using equation (3) and the monotonicity of u', this equation implies

$$\psi(h,b) = 1-\lambda^*,$$

or, replacing ψ using equation (9),

$$\frac{1-\lambda-\pi(1-h)c_1^* + b}{1-\pi} = 1-\lambda^*.$$

Solving for b yields

$$b = (1-\pi)\left(1-\lambda^* - \frac{1-\lambda-\pi(1-h)c_1^*}{1-\pi}\right)$$
$$= (1-\pi)\left(\hat{h}_{NB}(h,1) - \lambda^*\right),$$

as desired. □

Proposition 3. ***(Decentralized allocation in region 1)*** *There exists $\lambda_1^e < \lambda^*$ such that, when $\mu \le \mu_1$, the bank is bailed out if and only if $\lambda > \lambda_1^e$. In this region, the bank sets $h^e = 0$, patient investors do not run ($y^e = 2$), and the equilibrium bailout payment is*

$$b^e = \lambda - \lambda^* + \lambda^*\pi c_1^* \quad > \; b^*.$$

Proof. In states where the bank is bailed out, it will set its initial bail-in h either to zero or to the lowest value that prevents a run, $\underline{h}$. The cutoff μ_1 is defined so that $\mu \leq \mu_1$ implies $\underline{h} = 0$; it follows immediately that the bank will set $h^e = 0$ in these states and that patient investors do not run ($y^e = 2$). ix or

$$b^e = (1 - \pi c_1^*)(1 - \lambda^*) - (1 - \pi c_1^*)\frac{R}{(1 - \pi)c_2^*}(1 - \lambda - \pi c_1^*).$$

Using the resource constraint in equation (3) and regrouping terms yields

$$b^e = (1 - \lambda^*) - (1 - \lambda^*)\pi c_1^* - (1 - \lambda - \pi c_1^*)$$

or

$$b^e = \lambda - \lambda^* + \lambda^* \pi c_1^*,$$

as stated in the proposition. The planner's bailout b^* is shown in Proposition 1 to equal $\lambda - \lambda^*$. Since λ^*, π, and c_1^* are all strictly positive, the decentralized bailout is strictly larger than b^*.

What remains is to be shown that (i) there exists a cutoff λ_1^e such that the bank is bailed out if and only if $\lambda > \lambda_1^e$ and (ii) this cutoff is below the efficient level λ^*. In states where the bank is not bailed out, it will set $h = \hat{h} = \lambda$. The bank will choose $h = 0$, and hence be bailed out, if and only if doing so yields higher expected utility, that is,[3]

$$\pi \underbrace{u\left(c_1^*\right)}_{h=0} + (1-\pi) \underbrace{u\left((1-\lambda^*)c_2^*\right)}_{\text{bailed out}} > \pi \underbrace{u\left((1-\lambda)c_1^*\right)}_{h=\lambda} + (1-\pi) \underbrace{u\left((1-\lambda)c_2^*\right)}_{\text{not bailed out}}$$

Using the form of the utility function in equation (1), we can factor out the $(1 - \lambda)$ term on the right-hand side,

$$\pi\ u\left(c_1^*\right) + (1-\pi)u\big((1-\lambda^*)c_2^*\big) > (1-\lambda)^{1-\gamma}\big(\pi u\left(c_1^*\right) + (1-\pi)u\left(c_2^*\right)\big)$$

and solve for

$$\lambda > 1 - \left(\frac{\pi\ u\left(c_1^*\right) + (1-\pi)u\left((1-\lambda^*)c_2^*\right)}{\pi u\left(c_1^*\right) + (1-\pi)u\left(c_2^*\right)}\right)^{\frac{1}{1-\gamma}} \equiv \lambda_1^e.$$

To compare λ_1^e with λ^*, we use the explicit solution to the planner's problem,

$$c_1^* = \frac{1}{\pi + (1-\pi)R^{\frac{1-\gamma}{\gamma}}} \quad \text{and} \quad c_2^* = \frac{R^{\frac{1}{\gamma}}}{\pi + (1-\pi)R^{\frac{1-\gamma}{\gamma}}} \tag{27}$$

to obtain

$$\lambda_1^e = 1 - \left(\frac{\pi + (1-\pi)(1-\lambda^*)^{1-\gamma}R^{\frac{1-\gamma}{\gamma}}}{\pi + (1-\pi)R^{\frac{1-\gamma}{\gamma}}}\right)^{\frac{1}{1-\gamma}}$$

[3]The inequality is strict because we assume the bank chooses the larger bail-in if it is exactly indifferent.

Our assumption in equation (8) implies $\lambda^* > 0$ and, therefore,

$$\begin{aligned}
\lambda_1^e &< 1 - \left(\frac{\pi(1-\lambda^*)^{1-\gamma} + (1-\pi)(1-\lambda^*)^{1-\gamma} R^{\frac{1-\gamma}{\gamma}}}{\pi + (1-\pi) R^{\frac{1-\gamma}{\gamma}}} \right)^{\frac{1}{1-\gamma}} \\
&= 1 - \left((1-\lambda^*)^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \\
&= \lambda^*,
\end{aligned}$$

as desired. □

Proposition 4. ***(Decentralized allocation in region 2)*** *There exist $\mu_2 > \mu_1$ and $\lambda_2^e < \lambda^*$ such that, when $\mu_1 < \mu < \mu_2$, the bank is bailed out if and only if $\lambda > \lambda_2^e$. In this case, the bank sets $h^e = \underline{h} > 0$, patient investors do not run ($y^e = 2$), and the equilibrium bailout payment is*

$$b^e = \lambda - \lambda^* + (\lambda^* - \underline{h})\, \pi c_1^* \quad > \; b^*.$$

Proof. When $\mu > \mu_1$, $\underline{h}$ is strictly positive. In states where it is bailed out, the bank must choose between setting $h = \underline{h}$ to prevent a run and setting $h = 0$ and provoking a run. If will choose $h = \underline{h}$ if

$$\begin{aligned}
&\pi \underbrace{u\big((1-\underline{h})c_1^*\big)}_{h=\underline{h}} + (1-\pi) \underbrace{u\big((1-\lambda^*)c_2^*\big)}_{\text{all patient}} \\
&\qquad \geq \pi \underbrace{u\,(c_1^*)}_{h=0} + (1-\pi) \underbrace{\Big(\pi u\big((1-\lambda^*)c_1^*\big) + (1-\pi) u\big((1-\lambda^*)c_2^*\big)\Big)}_{\text{mix of impatient and patient}}
\end{aligned} \tag{28}$$

The definition of $\underline{h}$ in equation (17) implies

$$(1-\underline{h})c_1^* = (1-\lambda^*)c_2^*, \tag{29}$$

so we can write the previous inequality as

$$u\big((1-\lambda^*)c_2^*\big) \geq \pi u\,(c_1^*) + (1-\pi)\Big(\pi u\big((1-\lambda^*)c_1^*\big) + (1-\pi)u\big((1-\lambda^*)c_2^*\big)\Big)$$

Using the form of the utility function in equation (1), we have

$$(1-\lambda^*)^{1-\gamma} u(c_2^*) \geq \pi u\,(c_1^*) + (1-\pi)(1-\lambda^*)^{1-\gamma}\Big(\pi u(c_1^*) + (1-\pi)u(c_2^*)\Big)$$

or, bearing in mind that $\gamma > 1$ and $u(\cdot) < 0$,

$$(1-\lambda^*)^{1-\gamma} \leq \frac{u(c_1^*)}{(2-\pi)u(c_2^*) - (1-\pi)u(c_1^*)}.$$

Again using $\gamma > 1$, this expression can be written as

$$1 - \lambda^* \geq \left(\frac{u(c_1^*)}{(2-\pi)u(c_2^*) - (1-\pi)u(c_1^*)} \right)^{\frac{1}{1-\gamma}}$$

Next, we use equation (8) to replace λ^* on the left-hand side and equation (27) to replace c_1^* and c_2^* on the right-hand side, we have

$$\frac{1}{\mu^{\frac{1}{\gamma}} c_1^*} \geq \left(\frac{1}{(2-\pi)R^{\frac{1-\gamma}{\gamma}} - (1-\pi)} \right)^{\frac{1}{1-\gamma}}$$

or

$$\mu^{\frac{1}{\gamma}} c_1^* \leq \left((2-\pi)R^{\frac{1-\gamma}{\gamma}} - (1-\pi) \right)^{\frac{1}{1-\gamma}}$$

or

$$\mu \leq R\,(c_1^*)^{-\gamma} \underbrace{\left((2-\pi) - (1-\pi)R^{\frac{\gamma-1}{\gamma}} \right)^{\frac{\gamma}{1-\gamma}}}_{>1} \equiv \mu_2.$$

Given $\mu_1 = (c_1^*)^{-\gamma}$, the expression above shows $\mu_2 > \mu_1$ holds.

The argument above establishes that when $\mu \in (\mu_1, \mu_2]$, the bank will set $h = \underline{h} > 0$ and patient investors will not run ($y^e = 2$) in states where the is bailed out. Substituting these values into the expression for $b(h, y)$ in equation (14) yields

$$b^e = (1 - \pi c_1^*) \left(1 - \frac{R}{c_2^*} \frac{1 - \lambda - \pi(1 - \underline{h})c_1^*}{1 - \pi} - \lambda^* \right).$$

We can rewrite this expression as

$$b^e = (1 - \pi c_1^*)(1 - \lambda^*) - (1 - \pi c_1^*) \frac{R}{(1-\pi)c_2^*} (1 - \lambda - \pi(1 - \underline{h})c_1^*),$$

or, using the resource constraint in equation (3) and regrouping terms,

$$b^e = (1 - \lambda^*) - (1 - \lambda^*)\pi c_1^* - (1 - \lambda) + \pi(1 - \underline{h})c_1^*),$$

which simplifies to

$$b^e = \lambda - \lambda^* + (\lambda^* - \underline{h})\pi c_1^*.$$

Using $b^* = \lambda - \lambda^*$ and $\underline{h} < \lambda^*$, we have $b^e > b^*$.

What remains is to show that (i) there exists a cutoff λ_2^e such that the bank is bailed out if and only if $\lambda > \lambda_2^e$ and (ii) this cutoff is below the efficient level λ^*. The bank will choose $h = \underline{h}$, and hence be bailed out, rather than setting $h = \hat{h} = \lambda$ if and only if

$$\pi \underbrace{u\left((1-\underline{h})c_1^*\right)}_{h=\underline{h}} + (1-\pi)\underbrace{u\left((1-\lambda^*)c_2^*\right)}_{\text{bailed out}} > \pi \underbrace{u\left((1-\lambda)c_1^*\right)}_{h=\lambda} + (1-\pi)\underbrace{u\left((1-\lambda)c_2^*\right)}_{\text{not bailed out}}$$

Using equation (29) and the form of the utility function in equation (1), we can write this inequality as

$$(1-\lambda^*)^{1-\gamma} u\left(c_2^*\right) > (1-\lambda)^{1-\gamma}\big(\pi u\left(c_1^*\right) + (1-\pi)u\left(c_2^*\right)\big)$$

or, bearing in mind that $\gamma > 1$ and $u(\cdot) < 0$,

$$1-\lambda < (1-\lambda^*)\left(\pi \frac{u(c_1^*)}{u(c_2^*)} + (1-\pi)\right)^{\frac{1}{\gamma-1}}$$

or

$$\lambda > 1 - \underbrace{(1-\lambda^*)\left(\pi \frac{u(c_1^*)}{u(c_2^*)} + (1-\pi)\right)^{\frac{1}{\gamma-1}}}_{<1} \equiv \lambda_2^e.$$

Note that $\lambda_2^e < 1 - (1-\lambda^*) = \lambda^*$ is immediate from the expression above. □

Proposition 5. ***(Decentralized allocation in region 3)*** *There exists $\lambda_3^e < \lambda^*$ such that, when $\mu > \mu_2$, the bank is bailed out if and only if $\lambda > \lambda_3^e$. In this case, the bank sets $h^e = 0$, patient investors run ($y^e = 1$), and the equilibrium bailout payment is*

$$b^e = \lambda - \lambda^* + \lambda^* \pi c_1^* + (1-\lambda^*)\pi(c_1^* - 1).$$

Proof. When $\mu > \mu_2$, the proof of Proposition 4 shows that, in states where the bank is bailed out, the inequality in equation (28) is reversed, so the bank will set $h^e = 0$ and patient investors will run ($y^e = 2$). Substituting these values into the expression for $b(h, y)$ in equation (14) yields

$$b^e = (1-\pi)\left(1 - \frac{1-\lambda-\pi c_1^*}{1-\pi} - \lambda^*\right)$$

$$= \lambda - \lambda^* + \pi\big(c_1^* - (1-\lambda^*)\big)$$

$$= \lambda - \lambda^* + \lambda^* \pi c_1^* + \underbrace{(1-\lambda^*)\pi(c_1^* - 1)}_{\text{extra due to run}}.$$

This expression makes clear that b^e is strictly greater than $b^* = \lambda - \lambda^*$.

What remains is to be shown that (i) there exists a cutoff λ_3^e such that the bank is bailed out if and only if $\lambda > \lambda_1^e$ and (ii) this cutoff is below the efficient level λ^*. The bank will set $h = 0$ and be bailed out rather than choosing $h = \lambda$ if and only if

$$\pi \underbrace{u(c_1^*)}_{h=0} + (1-\pi)\underbrace{\Big(\pi u\big((1-\lambda^*)c_1^*\big) + (1-\pi)u\big((1-\lambda^*)c_2^*\big)\Big)}_{\text{mix of impatient and patient; bailed out}}$$

$$> \pi \underbrace{u\left((1-\lambda)c_1^*\right)}_{h=\lambda} \quad + \quad (1-\pi) \underbrace{u\left((1-\lambda)c_2^*\right)}_{\text{all patient; no bailout}}$$

Using the form of the utility function in equation (1), we can write this inequality as

$$\pi u(c_1^*) + (1-\lambda^*)^{1-\gamma}(1-\pi)\big(\pi u(c_1^*) + (1-\pi)u(c_2^*)\big) > (1-\lambda)^{1-\gamma}\big(\pi u(c_1^*) + (1-\pi)u(c_2^*)\big)$$

or

$$(1-\lambda)^{1-\gamma} > \frac{\pi u(c_1^*) + (1-\lambda^*)^{1-\gamma}(1-\pi)\big(\pi u(c_1^*) + (1-\pi)u(c_2^*)\big)}{\pi u(c_1^*) + (1-\pi)u(c_2^*)}$$

or, since $\gamma > 1$,

$$(1-\lambda) < \left(\frac{\pi u(c_1^*) + (1-\pi)u(c_2^*)}{\pi u(c_1^*) + (1-\lambda^*)^{1-\gamma}(1-\pi)\big(\pi u(c_1^*) + (1-\pi)u(c_2^*)\big)}\right)^{\frac{1}{\gamma-1}}$$

or

$$\lambda > 1 - \left(\frac{\pi u(c_1^*) + (1-\pi)u(c_2^*)}{\pi u(c_1^*) + (1-\lambda^*)^{1-\gamma}(1-\pi)\big(\pi u(c_1^*) + (1-\pi)u(c_2^*)\big)}\right)^{\frac{1}{\gamma-1}} \quad \equiv \lambda_3^e.$$

To compare λ_3^e with the efficient bailout cutoff λ^*, we use the fact that $\mu > \mu_2$ implies the inequality in equation (28) is reversed, which implies

$$\lambda_3^e < 1 - \left(\frac{\pi u(c_1^*) + (1-\pi)u(c_2^*)}{\pi u\big((1-\underline{h})c_1^*\big) + (1-\pi)u\big((1-\lambda^*)c_2^*\big)}\right)^{\frac{1}{\gamma-1}}.$$

Using equation (29) to replace $\underline{h}$, we have

$$\lambda_3^e < 1 - \left(\frac{\pi u(c_1^*) + (1-\pi)u(c_2^*)}{(1-\lambda^*)^{1-\gamma}u(c_2^*)}\right)^{\frac{1}{\gamma-1}}.$$

$$= 1 - \left(\pi\frac{u(c_1^*)}{u(c_2^*) + (1-\pi)}\right)^{\frac{1}{\gamma-1}}(1-\lambda^*)$$

$$< 1 - (1-\lambda^*) \quad = \lambda^*.$$

We have, therefore, established that the decentralized bailout cutoff λ_3^e is below the efficient cutoff λ^*, as desired. □

Proposition 6. ***(Optimal policy in region 1)*** *If $\mu \leq \mu_1$, then $D^* = [h_1^*, 1]$ with $h_1^* > 0$.*

We begin with two lemmas that establish technical properties of (i) withdrawal behavior and bailouts as functions of the bank's choice of initial bail-in h, and (ii) the solution to the regulator's choice problem.

Lemma 2. *The function $y(h)$ in equation (15) is weakly increasing. The composite function $b(h, y(h))$ defined by equations (14) and (15) is decreasing in h and is strictly decreasing whenever $b(h, y(h)) > 0$.*

Proof. For the first part of the lemma, equation (13) shows that $\hat{h}_{NB}(h, 2)$ is strictly decreasing in h. The right-hand side of the inequality in equation (15) is, therefore, weakly increasing in h, while the left-hand side is strictly decreasing. Moreover, $\lambda^* < 1$ implies $y(h) = 2$ will always hold for h sufficiently close to 1. If follows that either (i) $y(h) = 2$ for all $h \in [0, 1]$ or (ii) $y(h) = 1$ for $h < x$ and $y(h) = 2$ for $h \geq x$ for some $x \in (0, 1)$; in both cases, $y(h)$ is weakly increasing. Intuitively, a larger bail-in always decreases the incentive for patient investors to run.

For the second part of the lemma, we first show $b(h, y)$ is decreasing in h holding y fixed. For either value of y, equation (13) shows $\hat{h}_{NB}(h, y)$ is strictly decreasing in h. Equation (14) then shows that for any $h' > h$, we have $b(h', y) \leq b(h, y)$ for any y, with strict inequality if $b(h, y) > 0$. Intuitively, if the bank paid less to the investors who have already withdrawn, it receives a smaller bailout.

We next show $b(h, y)$ is decreasing in y. Using $c_2^* < R$ in equation (13) shows that $\hat{h}_{NB}(h, y)$ decreases as we move from $y = 1$ to $y = 2$ for any h. Using this fact in equation (14), together with $c_1^* > 1$, implies we have $b(h, 2) \leq b(h, 1)$ for any h. Intuitively, the bank receives a smaller bailout if there is no run.

Combining these two results with the first part of the lemma shows that for any $h' > h$, we have

$$b\left(h, y(h)\right) \geq b\left(h', y(h)\right) \geq b\left(h', y(h')\right),$$

where the first inequality is strict if $b\left(h, y(h)\right) > 0$. Intuitively, a higher bail-in h leaves the bank with more resources and may, in addition, prevent a run. Both of these effects decrease the bailout payment it receives. □

Lemma 3. *For any closed delegation set D, (i) a solution to the bank's maximization problem (18) exists for every $\lambda \in \Lambda$, and (ii) there exists $\lambda_D^e \in \Lambda$ such that the bank is bailed out in this solution if and only if $\lambda > \lambda_D^e$.*

Proof. Part (i): For any fixed $\lambda \in \Lambda$, the function $W_B(h; \lambda)$ defined in equation (16) is continuous in h except at points where $y(h)$ changes value. Lemma 2 shows that $y(h)$ changes value at most once as h increases from 0 to 1: if $y(0) = 1$, the value changes to $y(h) = 2$ when h reaches the point where withdrawing early is no longer a strictly dominant strategy. As a result, W_B is an upper semi-continuous function of h on the unit interval and, therefore, attains a maximum on any compact subset D.

Part (ii): Suppose the bank is not bailed out under its optimal choice $h_D^e(\lambda)$ for some λ. We will show that the bank is also not bailed out under its optimal choice for any $\lambda' < \lambda$. It then follows that the set of ϕ for which the bank is not bailed out is an interval of the form $[0, \lambda_D^e]$ for some $\lambda_D^e \in \Lambda$.

Equations (13) and (15) show that when λ decreases, the set of h that lead to a bailout becomes weakly smaller. If there is no choice $h \in D$ that leads to a bailout for realization λ, therefore, the same is true for any $\lambda' < \lambda$ and the result is established. If there are some $h \in D$ that lead to a bailout in state λ, let $\hat{h}(\lambda)$ denote the best such choice. Since h_D^e is an optimal choice, we clearly have

$$W_B(h_D^e(\lambda); \lambda) \geq W_B(\hat{h}(\lambda); \lambda).$$

Now consider any $\lambda' < \lambda$. It is straightforward to show that $W_B(h; \lambda)$ is non-increasing in λ (holding h fixed) so we have

$$W_B(h_D^e(\lambda'); \lambda') \geq W_B(h_D^e(\lambda); \lambda') \geq W_B(h_D^e(\lambda); \lambda).$$

What remains to be shown is that $h_D^e(\lambda')$ does not lead to a bailout. Let $\hat{h}(\lambda')$ denote the best choice that does lead to a bailout.[4] Because the set of h that lead to a bailout is increasing in λ, $\hat{h}(\lambda')$ would have also led to a bailout under the original realization. Moreover, when the bank is bailed out, its payoff is independent of λ, so we have

$$W_B(\hat{h}(\lambda'); \lambda') = W_B(\hat{h}(\lambda'); \lambda) \leq W_B(\hat{h}(\lambda); \lambda).$$

Combining the above inequalities shows $W_B(h_D^e(\lambda'); \lambda') \geq W_B(\hat{h}(\lambda'); \lambda')$, meaning the bank is not bailed out under its optimal choice for λ' and we have established the result. □

Proof of Proposition 6. The proof is divided into two steps. We first show the optimal delegation set must be an interval of the form $[h_1, 1]$ for some $h_1 \geq 0$. We then show this lower bound is strictly positive.

Step (i): Show $D^* = [h_1, 1]$ for some $h_1 \geq 0$.

Given any compact delegation set D, define a new set $\hat{D} \equiv [h_1, 1]$ where h_1 is the smallest element of D. To establish this step, we show that $\hat{D}$ weakly dominates D, that is, $\mathcal{W}(\hat{D}) \geq \mathcal{W}(D)$. It then follows that the largest optimal delegation set D^* must also have the form.

Because $\hat{D}$ contains D, the bank's optimized payoff must be at least as high,

$$W_B\left(h_{\hat{D}}^e(\lambda); \lambda\right) \geq W_B\left(h_D^e(\lambda); \lambda\right) \quad \text{for all } \lambda \in \Lambda. \tag{30}$$

The regulator's payoff equals W_B plus the cost of any bailout payment. To establish that the regulator's payoff is also at least as high, we will show that the bailout associated with $h_{\hat{D}}^e(\lambda)$ is no larger than the bailout associated with $h_D^e(\lambda)$ for all λ.

Consider first any λ such that $h_D^e(\lambda) = h_1$. In these cases, the bank's choice of h cannot decrease when we move to policy $\hat{D}$. Using the second part of Lemma 2, therefore, the bailout payment cannot increase, that is

$$b\left(h_{\hat{D}}^e(\lambda), 2\right) \leq b\left(h_D^e(\lambda), 2\right) \text{ for any } \lambda \text{ such that } h_D^e(\lambda) = h_1. \tag{31}$$

[4]If no such choice exists for realization λ', the bank is clearly not bailed out and the result is established.

Next consider any λ such that $h^e_D(\lambda) > h_1$. In these states, the bank is not bailed out under policy D. While the bank's optimal choice of bail-in h may decrease when we move to policy $\hat{D}$, the bailout must remain zero. To see why, suppose this were not true, that is, suppose the bank were bailed out following $h^e_{\hat{D}}(\lambda)$. Because $\mu \leq \mu_1$, we know $\underline{h}$ as defined in equation (17) is zero and no choice of bail-in will lead to a run. If the bank is being bailed out, therefore, it must be choosing the smallest element of $\hat{D}$, that is, $h^e_{\hat{D}}(\lambda) = h_1$. But h_1 was a feasible choice under policy D as well, which contradicts the fact that $h^e_D(\lambda) > h_1$ was chosen.[5] We thus have

$$b\left(h^e_{\hat{D}}(\lambda), 2\right) = b\left(h^e_D(\lambda), 2\right) = 0 \text{ for any } \lambda \text{ such that } h^e_D(\lambda) > h_1. \tag{32}$$

Combining equations (30) – (32) yields

$$W_R\left(h^e_{\hat{D}}(\lambda); \lambda\right) \geq W_R\left(h^e_D(\lambda); \lambda\right) \text{ for all } \lambda \in \Lambda. \tag{33}$$

Using the definition of $\mathcal{W}$ in equation (19), we then have $\mathcal{W}(\hat{D}) \geq \mathcal{W}(D)$, as desired.

Step (ii): Show $h^*_1 > 0$.

Because the optimal delegation set has the form $[h_{min}, 1]$, we can write the regulator's expected payoff as

$$\int_0^{h_1} W_R\left(h_1; \lambda\right) dF(\lambda) + \int_{h_1}^{\lambda^e_D} W_R\left(\lambda; \lambda\right) dF(\lambda) + \int_{\lambda^e_D}^{\bar{\lambda}} W_R\left(h_1; \lambda\right) dF(\lambda).$$

If the bank has a zero or small loss, it chooses the smallest allowable bail-in, h_1. When the loss is between h_1 and the bailout cutoff λ^e_D, the bank sets $h = \lambda$ and is not bailed out. When the loss is larger than λ^e_D, the bank chooses the smallest allowable bail-in and is bailed out. The bailout cutoff also depends on h_1 and can be shown in this case to be

$$\lambda^e_D\left(h_1\right) = 1 - \left(\frac{\pi u\left((1-h_1)c^*_1\right) + (1-\pi)u\left(\phi^* c^*_2\right)}{\pi u\left(c^*_1\right) + (1-\pi)u\left(c^*_2\right)}\right)^{\frac{1}{1-\gamma}}. \tag{34}$$

It is straightforward to show this cutoff is increasing in h_1. When the minimum bail-in is larger, being bailed out is less attractive to the bank and the set of states in which a bailout occurs shrinks.

Because the distribution F may put positive probability on $\lambda = 0$, it is useful to rewrite the regulator's payoff using the density function f. Letting $z \geq 0$ denote the probability of $\lambda = 0$, we have

$$z\, W_R\left(h_1; 0\right) + \int_0^{h_1} W_R\left(h_1; \lambda\right) f(\lambda) d\lambda + \int_{h_1}^{\lambda^e_D} W_R\left(\lambda; \lambda\right) f(\lambda) d\lambda \tag{35}$$
$$+ \int_{\lambda^e_D}^{\bar{\lambda}} W_R\left(h_1; \lambda\right) f(\lambda) d\lambda.$$

[5] Recall that, if the bank were indifferent between $h = h_1$ and $h = h^e_D(\lambda) > h_1$, it would have chosen the larger bail-in under our tie-breaking rule.

Investors never have an incentive to run when $\mu < \mu_1$, meaning $y(h) = 2$ holds for all h. In this case, the function $W_R(h; \lambda)$ is continuous in h and is differentiable for all λ except the bailout cutoff λ_D^e. We can, therefore, write the slope of the regulator's expected payoff in equation (35) with respect to h_1 as

$$z \frac{dW_R}{dh}(h_1; 0) + \int_0^{h_1} \frac{dW_R}{dh}(h_1; \lambda) f(\lambda) d\lambda \tag{36}$$
$$+ \Big[W_R(\lambda_D^e; \lambda_d^e) - W_R(h_1, \lambda_D^e) \Big] f(\lambda_D^e) \frac{d\lambda_D^e}{dh_1} + \int_{\lambda_D^e}^{\bar{\lambda}} \frac{dW_R}{dh}(h_1; \lambda) f(\lambda) d\lambda.$$

The first two terms in this expression capture the cost of raising h_1: it increases the distortion in states where the bank has no loss or only a small loss. The last two terms capture the benefit of increasing h_1: it shrinks the set of states where the bank is bailed out and increases the bail-in the bank must use in those states. To evaluate this slope at $h_1 = 0$, we write out the first term as

$$\frac{dW_R}{dh}(h_1; 0) = \pi c_1^* \left(-u'\left((1-h_1)c_1^*\right) + Ru'\left(\frac{R}{1-\pi}(1 - \pi(1-h_1)c_1^*) \right) \right).$$

Evaluating this term at $h_1 = 0$ yields

$$\frac{dW_R}{dh}(0; 0) = \pi c_1^* \Big(-u'(c_1^*) + Ru'(c_2^*) \Big) = 0. \tag{37}$$

In other words, as h_1 increases from zero, the cost of the distortion when the bank has no loss is second-order because the bank was at an unconstrained optimum. The second term in equation (36) also vanishes when $h_1 = 0$. The third and fourth terms, in contrast, remain strictly positive. It follows that the regulator's objective function is strictly increasing at $h_1 = 0$ and, therefore, the optimal choice h_1^* is strictly positive. □

Proposition 7. ***(Optimal policy in regions 2 and 3)*** *If $\mu > \mu_1$ then $D^* = [h_0, \underline{h} - \varepsilon] \cup [h_1, 1]$ with $0 \leq h_0^* \leq \underline{h} \leq h_1^* < 1$. Moreover, at least one of $h_0^* > 0$ and $h_1^* > \underline{h}$ holds with strict inequality.*

Proof. We follow a similar approach to that in the proof of Proposition 6. Given any delegation set D, we first define another set $\hat{D}$ that contains D and is the union of two intervals, as in the statement of the proposition. We show that $\hat{D}$ generates a payoff at least as high as D and, therefore, the optimal delegation set must have this form. We then establish that at least one of the inequalities in the proposition is strict.

Step 1: Define the new set $\hat{D}$.

Given any D, let h_1 denote its smallest element satisfying $h_1 \geq \underline{h}$. In other words, h_1 is the smallest bail-in the bank can choose when it is in the bailout region without causing a run. Because $\mu > \mu_1$, we have $\underline{h} > 0$ and, hence, h_1 is strictly positive as well. Let h_0 denote the smallest overall element of D. Define

$$\hat{D} = [h_0, \underline{h}) \cup [h_1, 1]. \tag{38}$$

Note that $\hat{D}$ contains the original delegation set D by construction. It consists of two disjoint intervals. All h in the lower interval would cause a run if chosen when the bank is in the bailout region, while all h in the upper interval would prevent a run. In states where the bank is bailed out, it will choose the smallest element of one of these two intervals, that is, either h_0 or h_1.

Step 2: Show $\mathcal{W}(\hat{D}) \geq \mathcal{W}(D)$.

Consider first an intermediate delegation set,

$$D' = D \cup [h_1, 1] .$$

That is, suppose we add to D only those bail-in choices that lie above h_1. The argument that moving from D to D' cannot decrease the regulator's payoff follows Step 1 in the proof of Proposition 6 closely. Since D' contains D, the bank's optimized payoff must be as least as high

$$W_B\left(h^e_{D'}(\lambda); \lambda\right) \geq W_B\left(h^e_D(\lambda); \lambda\right) \quad \text{for all } \lambda \in \Lambda. \tag{39}$$

In all states $\lambda > \lambda^e_D$, the bank is bailed out and sets $h^e_D(\lambda)$ to either h_0 or h_1. Since the additions in moving to D' are all larger than both h_0 and h_1, Lemma 2 shows that the bailout received by the bank in these states cannot increase,

$$b\Big(h^e_{D'}(\lambda), y\left(h^e_{D'}(\lambda)\right)\Big) \leq b\Big(h^e_D(\lambda), y\left(h^e_D(\lambda)\right)\Big) \text{ for all } \lambda < \lambda^e_D. \tag{40}$$

For $\lambda \leq \lambda^e_D$, the bank's choice of h may either increase or decrease when we move to D'. However, since the bank is not bailed out under policy D, it must also not be bailed out under its optimal choice from D'. To see why, suppose it were bailed out under D'. Then $h^e_{D'}(\lambda)$ must equal either h_0 or h_1. But both of these options were available under D as well, contradicting the fact that the bank did not choose them and receive a bailout under policy D. We therefore have

$$b\Big(h^e_{D'}(\lambda), y\left(h^e_{D'}(\lambda)\right)\Big) = b\Big(h^e_D(\lambda), y\left(h^e_D(\lambda)\right)\Big) = 0 \text{ for all } \lambda \leq \lambda^e_D. \tag{41}$$

Combining equations (39) – (41) with the definition of $\mathcal{W}$ in equation (19) shows that we have $\mathcal{W}(D') \geq \mathcal{W}(D)$, that is, moving to delegation set D' weakly increases the regulator's payoff.

Next, we show that moving from D' to $\hat{D}$ in equation (38) also weakly increases the regulator's payoff. Note that $\hat{D}$ contains D' by construction, so we have the usual result that the bank's optimized payoff cannot decrease

$$W_B\left(h^e_{\hat{D}}(\lambda); \lambda\right) \geq W_B\left(h^e_{D'}(\lambda); \lambda\right) \quad \text{for all } \lambda \in \Lambda. \tag{42}$$

All that remains is to show that the bailout payment to the bank does not increase for any λ. Moving from D' to $\hat{D}$ adds choices of h that will cause a run if chosen when the bank is in the bailout region. For $\lambda > \lambda^e_{D'}$, any $h \in (h_0, \underline{h})$ is strictly inferior to choosing h_0. If the bank is going to suffer a run, it would prefer to set the smallest bail-in possible. Since h_0 was also available under D' and was not chosen, it must not be optimal under $\hat{D}$ either and the bank's optimal choice remains unchanged,

$$h^e_{\hat{D}}(\lambda) = h^e_{D'}(\lambda) \quad \text{for all } \lambda > \lambda^e_{D'}. \tag{43}$$

When $\lambda \leq \lambda^e_{D'}$, the bank is not bailed out under D'. In this case, the bank must not be bailed out following its optimal choice under $\hat{D}$ either. To see why, suppose it were bailed out under $\hat{D}$. Then its optimal choice $h^e_{\hat{D}}$ must be either h_0 or h_1. But both of these options were available under policy D', contradicting the fact that $\lambda \leq \lambda^e_{D'}$. Therefore, we have

$$b\Big(h^e_{\hat{D}}(\lambda), y\left(h^e_{\hat{D}}(\lambda)\right)\Big) = b\Big(h^e_{D'}(\lambda), y\left(h^e_{D'}(\lambda)\right)\Big) = 0 \text{ for all } \lambda \geq \lambda^e_{D'}. \tag{44}$$

Equations (42) - (44) imply we have

$$\mathcal{W}(\hat{D}) \geq \mathcal{W}\left(D'\right) \geq \mathcal{W}(D),$$

as desired. Together, steps (i) and (ii) show that the optimal delegation set must have the form in equation (38). As discussed in the main text, we restrict the regulator to choose a closed set to ensure the bank's optimization problem has a solution in all states. If $h_1 > \underline{h}$, the bank may want to choose the bail-in h closest to $\underline{h}$ in some states where it is not bailed out, but no such closest number exists in $\hat{D}$. To avoid this technical complication, we approximate the form in equation (38) by

$$D^*_\varepsilon = [h_0, \underline{h} - \varepsilon] \cup [\underline{h}, 1]\,,$$

and state our results in terms of the limiting case where ε approaches zero.

Step 3: Show at least one of $h^*_0 > 0$ and $h^*_1 > \underline{h}$ holds with strict inequality.

We establish the final step by contradiction. Suppose both $h^*_0 = 0$ and $h^*_1 = \underline{h}$ held. Then, taking the limiting case where $\varepsilon \to 0$, D^* would be all of the unit interval, as studied in Section 4. We will show that increasing one or both of these lower bounds would raise welfare, contradicting the claim that $D^* = [0, 1]$ is optimal. We break the analysis into cases based on the public sector's marginal cost of funds.

Case (i): $\mu_1 < \mu < \mu_2$

In this case, Proposition 4 establishes that $h^e(\lambda) = \underline{h} > 0$ for all $\lambda > \lambda^e$ when $D = [0, 1]$. In other words, in states where the bank is bailed out, it will choose the smallest bail-in that prevents a run. Moreover, $\mu < \mu_2$ implies this preference is strict, meaning the regulator can increase h_1 slightly above $\underline{h}$ and the bank will still prefer choosing h_1 over setting $h = 0$ and experiencing a run in those states where it is bailed out. Within this neighborhood, and keeping h^*_0 fixed at zero, we can write the regulator's expected payoff as a function of $h_1 \geq \underline{h}$

as

$$\int_0^{\underline{h}-\varepsilon} W_R(\lambda;\lambda)\, dF(\lambda) + \int_{\underline{h}-\varepsilon}^{\hat{\lambda}} W_R(\underline{h}-\varepsilon;\lambda)\, dF(\lambda) + \int_{\hat{\lambda}}^{h_1} W_R(h_1;\lambda)\, dF(\lambda) \qquad (45)$$

$$+ \int_{h_1}^{\lambda_D^e} W_R(\lambda;\lambda)\, dF(\lambda) + \int_{\lambda_D^e}^{\bar{\lambda}} W_R(h_1;\lambda)\, dF(\lambda)$$

where $\hat{\lambda}$ is the state where the bank is indifferent between h_1 and $\underline{h}-\varepsilon$, assuming it is not bailed out in either case,

$$W_B\left(h_1;\hat{\lambda}\right) = W_B\left(\underline{h}-\varepsilon;\hat{\lambda}\right), \qquad (46)$$

and λ_D^e depends on h_1 as shown in equation (34) above. The first four terms in equation (45) correspond to states where the bank is not bailed out. When λ is less than $\underline{h}-\varepsilon$, the bank chooses the efficient bail-in $h=\lambda$. For λ between $\underline{h}-\varepsilon$ and h_1, the efficient bail-in lies in the "hole" of the delegation set and the bank must either bail-in less ($\underline{h}-\varepsilon$) or more ($h_1$). Equation (46) defines the cutoff below which the bank prefers $\underline{h}-\varepsilon$ and above which it prefers h_1. Finally, when λ is larger than λ_D^e, the bank chooses h_1 and is bailed out.

Differentiating the objective function with respect to h_1 yields

$$\int_{\hat{\lambda}}^{h_1} \frac{dW_R}{dh}(h_1;\lambda)\, dF(\lambda) + \left[W_R(\lambda_D^e;\lambda_d^e) - W_R(h_1,\lambda_D^e)\right] f(\lambda_D^e)\frac{d\lambda_D^e}{dh_1} \qquad (47)$$

$$+ \int_{\lambda_D^e}^{\bar{\lambda}} \frac{dW_R}{dh}(h_1;\lambda)\, dF(\lambda).$$

The first term in equation (47) captures the cost of distorting the bail-in in those states where the bank is not bailed out, the efficient bail-in lies in the "hole" of the delegation set, and the bank ends up choosing h_1. This term is negative for all $h_1 > \underline{h}$. The second term captures the change in the set of states where the bank is bailed out. The term in square brackets is positive when h_1 is close to $\underline{h}$. Since λ_D^e is increasing in h_1, this second term is strictly positive. The third term captures the effect of increasing the bail-in above $\underline{h}$ in states where the bank is bailed out. This term is also strictly positive when h_1 is close to $\underline{h}$. Note that no $d\hat{\lambda}/dh_1$ term appears in the derivative because the payoff function is continuous at $\hat{\lambda}$.

Evaluating this derivative at $h_1 = \underline{h}$ and taking the limit as $\varepsilon \to 0$, we have $\hat{\lambda} \to \underline{h}$ and the first term in equation (47) becomes zero. Since the other two terms remain strictly positive, the derivative is strictly positive at $h_1 = \underline{h}$. If $h_0^* = 0$, therefore, the optimal value of h_1^* must be strictly positive.

Case (ii): $\mu > \mu_2$

In this case, Proposition 5 establishes $h^e(\lambda) = 0$ for all $\lambda > \lambda^e$ when $D = [0,1]$. In other words, in states where the bank is bailed out, it chooses no bail-in and investors run on the bank. Moreover, $\mu > \mu_2$ implies this preference is strict, meaning the the regulator can increase h_0 slightly above 0 and the bank will still choose the lowest possible bail-in and experience a run in those states where it is bailed out. Within this neighborhood, and keeping h_1^* fixed at $\underline{h}$, we can write the regulator's expected payoff as a function of $h_0 \geq 0$ as

$$z\, W_R\,(h_0; 0) + \int_0^{h_0} W_R\,(h_0; \lambda)\, f(\lambda) d\lambda + \int_{h_0}^{\lambda_D^e} W_R(\lambda; \lambda) f(\lambda) d\lambda \\ + \int_{\lambda_D^e}^{\bar{\lambda}} W_R\,(h_0; \lambda)\, f(\lambda) d\lambda$$

where f is the density function for $\lambda > 0$ and $z \geq 0$ is the probability of $\lambda = 0$. Note that this equation looks nearly identical to the objective in equation (35) in the proof of Proposition 6 above, only with h_0 replacing h_1. The difference between the two equations lies inside the W_R term for $\lambda > \lambda_D^e$, which now captures the fact that a run is occurring in these states. Despite this difference, the steps are identical to those following equation (35) and are omitted here. Following those steps shows that, when $h_1 = \underline{h}$, increasing h_0 above zero creates a first-order gain for the regulator in states where the bank is bailed out and has no first-order cost in states where the bank is sound. As a result, $h_0^* > 0$ must hold.

Case (iii): $\mu = \mu_2$

The final case is where the public sector's marginal cost of funds lies exactly on the boundary between the two previous cases. The analysis in Section 4.6 of the main text shows that, in this case, the bank is indifferent between setting $h = \underline{h} > 0$, which prevents a run, and setting $h = 0$, which provokes a run. We assume the bank chooses $h = \underline{h}$ in this situation, but increasing h_1^* even slightly above $\underline{h}$ would lead the bank to switch to $h = 0$ in states where it is bailed out. To increase the regulator's payoff in this case, therefore, we need to raise both h_0 and h_1 together in such a way that the bank continues to be willing to choose the higher of the two values.

Give some $h_1 \geq \underline{h}$, let $g(h_1)$ be the bail-in satisfying

$$\pi u\left((1 - h_1)c_1^*\right) + (1 - \pi) u\left((1 - \lambda^*)c_1^*\right) = \\ \pi u\big((1 - g(h_1))c_1^*\big) + (1 - \pi)\big[\pi u\left((1 - \lambda^*)c_1^*\right) + (1 - \pi) u\left((1 - \lambda^*)c_2^*\right)\big]$$

In other words, the bank is indifferent between setting h_1 with no run and setting $g(h_1)$ with a run. Using the form of the utility function in equation (1), we can solve this equation for

$$g\,(h_1) = 1 - \left((1 - h_1)^{1-\gamma} + (1 - \pi)\left(\frac{u((1 - \lambda^*)c_2^*)}{u((1 - \lambda^*)c_1^*)} - 1\right)\right)^{\frac{1}{1-\gamma}}. \qquad (48)$$

When $\mu = \mu_2$, we have $g(\underline{h}) = 0$. (This is effectively the definition of μ_2 from the proof of Proposition 4.) It is straightforward to show from equation (48) that $g(h_1)$ is strictly increasing and differentiable as h_1 increases above $\underline{h}$. When h_0 is set to $g(h_1)$, we can write the regulator's payoff as a function of h_1 as

$$z\, W_R\left(g(h_1);0\right) + \int_0^{g(h_1)} W_R\left(g(h_1);\lambda\right) f(\lambda) d\lambda + \int_{g(h_1)}^{\underline{h}-\varepsilon} W_R\left(\lambda;\lambda\right) f(\lambda) d\lambda$$
$$+ \int_{\underline{h}-\varepsilon}^{\hat{\lambda}} W_R\left(\underline{h}-\varepsilon;\lambda\right) f(\lambda) d\lambda + \int_{\hat{\lambda}}^{h_1} W_R\left(h_1;\lambda\right) f(\lambda) d\lambda + \int_{h_1}^{\lambda_D^e} W_R\left(\lambda;\lambda\right) f(\lambda) d\lambda$$
$$+ \int_{\lambda_D^e}^{\bar{\lambda}} W_R\left(h_1;\lambda\right) f(\lambda) d\lambda.$$

When the bank has zero loss or a small loss, it chooses the smallest allowable bail-in, $g(h_1)$. For λ between $g(h_1)$ and $\underline{h}-\varepsilon$, the bank is not bailed out and is able to choose the efficient bail-in, λ. For λ between $\underline{h}-\varepsilon$ and h_1 , the efficient bail-in lies in the hole of the delegation set and the bank will choose either $\underline{h}-\varepsilon$ or h_1. As in case (i) above, the cutoff state between these two choices, $\hat{\lambda}$, is given by equation (46). For λ between h_1 and λ_D^e, the bank is again able to choose the efficient bail-in, λ. Finally, for λ greater than λ_D^e, the bank chooses h_1 and prevents a run, as in case (i) above.

Differentiating this objective function with respect to h_1 yields

$$z \frac{dW_R}{dh}\left(g(h_1);0\right) g'(h_1) + \int_0^{g(h_1)} \frac{dW_R}{dh}\left(g(h_1);\lambda\right) g'(h_1) f(\lambda) d\lambda$$
$$+ \int_{\hat{\lambda}}^{h_1} \frac{dW_R}{dh}\left(h_1;\lambda\right) f(\lambda) d\lambda + \Big[W_R\left(\lambda_D^e;\lambda_d^e\right) - W_R\left(h_1,\lambda_D^e\right)\Big] f(\lambda_D^e) \frac{d\lambda_D^e}{dh_1}$$
$$+ \int_{\lambda_D^e}^{\bar{\lambda}} \frac{dW_R}{dh}\left(h_1;\lambda\right) f(\lambda) d\lambda.$$

We evaluate this derivative at $h_1 = \underline{h}$ and take the limit as $\varepsilon \to 0$, which implies $\hat{\lambda} \to \underline{h}$ and, hence, the third term in the derivative is zero. In addition, $g(\underline{h}) = 0$ implies the second term is zero and the derivative reduces to

$$z \underbrace{\frac{dW_R}{dh}\left(0;0\right)}_{=0} g'(\underline{h}) + \underbrace{\Big[W_R\left(\lambda_D^e;\lambda_D^e\right) - W_R\left(\underline{h},\lambda_D^e\right)\Big]}_{>0} f(\lambda_D^e) \underbrace{\frac{d\lambda_D^e}{dh_1}}_{>0}$$
$$+ \int_{\lambda_D^e}^{\bar{\lambda}} \underbrace{\frac{dW_R}{dh}\left(h_1;\lambda\right)}_{>0} f(\lambda) d\lambda. \quad > \quad 0.$$

The first term measures the first-order cost of distorting the choice in states where the bank has no loss, which is shown to be zero in equation (37) above. The second term captures the benefit of shrinking the set of states where the bail is bailed out and is strictly positive. The final term captures the benefit of increasing the bail-in in states where the bank is bailed out, which is also positive. As a result, the derivative is strictly positive when evaluated at $h_1 = \underline{h}$. The optimal policy must, therefore, have either $h_0^* > 0$, $h_1^* > \underline{h}$, or both. □