Supplemental Appendix for: "Should Central Banks Issue Digital Currency?"

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This appendix provides proofs of the propositions presented in the paper.

Proposition 1. *The economy with no digital currency has a unique equilibrium. There is a liquidity premium on deposits in this equilibrium if and only if* $\eta < \frac{\lambda_2 w(q^*)}{\bar{\gamma} - \frac{1}{\beta\theta}}$. *When this condition holds, the equilibrium interest rate is a strictly increasing function of* η.

Proof. Equations (13) and (14) immediately imply that any equilibrium is stationary, with m_t and r_t constant over time. We denote the equilibrium values by m^N and r^N, respectively. Because $\mu > \beta$, the properties of L imply that equation (13) has a unique solution for real money balances, $m^N \in (0, w(q^*)/\beta)$. Moreover, the equation implicitly defines m^N as a function of μ in this region, with

$$\frac{dm^N}{d\mu} = \frac{L\left(\frac{m^N}{\mu}\right) + \frac{m^N}{\mu} L'\left(\frac{m^N}{\mu}\right)}{L'\left(\frac{m^N}{\mu}\right)} < 0.$$

That is, equilibrium real money balances are strictly decreasing in the inflation rate μ.

We can write the market-clearing equation (14) for deposits as

$$\lambda_2 \frac{L^{-1}\left(\frac{1}{\beta(1+r)}\right)}{1+r} = \eta\left(\bar{\gamma} - \frac{1+r}{\theta}\right). \tag{A1}$$

The left-hand side of (A1) is the demand for deposits, and the right-hand side is the supply. Our assumptions imply that the left-hand side is a continuous, strictly increasing function of $1+r$ that starts below $\eta\bar{\gamma}$ and approaches $\lambda_2 w(q^*)$ as $1+r \to \beta^{-1}$. The demand for deposits becomes vertical (i.e., a correspondence) when $1+r = \beta^{-1}$, including all points greater than or equal to $\lambda_2 w(q^*)$. The right-hand side of (A1) starts at $\eta\bar{\gamma}$ and is a decreasing, linear function of $1+r$. It follows that equation (A1) has a unique solution, $1+r^N$, satisfying

$$1 + r^N \begin{Bmatrix} < \\ = \end{Bmatrix} \frac{1}{\beta} \quad \text{as} \quad \eta \begin{Bmatrix} < \\ \geq \end{Bmatrix} \frac{\lambda_2 w(q^*)}{\bar{\gamma} - \frac{1}{\beta\theta}}.$$

In the first case, we can differentiate through equation (A1) to obtain

$$\frac{\partial\left(1+r^{N}\right)}{\partial\eta}=-\frac{\frac{\beta}{\lambda_{2}}\left(1+r^{N}\right)^{2}\left(\bar{\gamma}-\frac{1+r^{N}}{\theta}\right)L'\left(A^{N}\right)}{L\left(A^{N}\right)+A^{N}L'\left(A^{N}\right)-\frac{\eta}{\lambda_{2}}\frac{\left(1+r^{N}\right)^{2}}{\theta}L'\left(A^{N}\right)}>0,$$

where

$$A^{N}\equiv\eta\left(\frac{1+r^{N}}{\lambda_{2}}\right)\left(\bar{\gamma}-\frac{1+r^{N}}{\theta}\right),$$

which shows that the equilibrium interest rate is strictly increasing in η in this region. □

Proposition 2. *Under a cash-like digital currency with $i>0$, the unique equilibrium allocation satisfies $e_1^C>m^N$, $q_1^C>q_1^N$ and $\left(q_2^C,\hat{\gamma}^C\right)=\left(q_2^N,\hat{\gamma}^N\right)$.*

Proof. Consider first an artificial economy with no physical currency, so that only digital currency is used in type 1 DM meetings. A type 1 buyer's real money balances, e_1, would satisfy the first-order condition

$$L\left(\frac{(1+i)}{\mu}e_{1}\right)=\frac{\mu}{\beta\left(1+i\right)},$$

which implicitly defines e_1 as a function of the interest rate $1+i$. The buyer's holding of real physical currency balances in the economy with no CBDC, m^N, is equal to this value of e_1 when $i=0$. Differentiating through this condition and solving yields

$$\frac{de_{1}}{d\left(1+i\right)}=-\frac{L\left(\frac{(1+i)}{\mu}e_{1}\right)+\frac{(1+i)}{\mu}e_{1}L'\left(\frac{(1+i)}{\mu}e_{1}\right)}{\frac{(1+i)^{2}}{\mu}L'\left(\frac{(1+i)}{\mu}e_{1}\right)}>0,$$

where the fact that the numerator is positive follows from our assumption that $AL(A)$ is strictly increasing in A. It follows that $e_1^C>m^N$ holds whenever $i>0$.

To see that the digital currency leads to an increase in DM production in type 1 meetings, note that the quantity produced satisfies

$$\alpha\frac{u'\left(q_{1}^{C}\right)}{w'\left(q_{1}^{C}\right)}+1-\alpha=\frac{\mu}{\beta\left(1+i\right)}<\frac{\mu}{\beta}=\alpha\frac{u'\left(q_{1}^{N}\right)}{w'\left(q_{1}^{N}\right)}+1-\alpha.$$

This inequality implies

$$\frac{u'\left(q_{1}^{C}\right)}{w'\left(q_{1}^{C}\right)}<\frac{u'\left(q_{1}^{N}\right)}{w'\left(q_{1}^{N}\right)},$$

which, in turn, implies $q_1^C>q_1^N$.

Finally, because a cash-like digital currency cannot be used in type 2 DM meetings, it will not be held by type 2 buyers. The quantities of deposits, investment, and type 2 DM production therefore remain unchanged. □

Proposition 3. *There exists $\bar{\nu} \in (0,1)$ such that a cash-like digital currency is desirable if and only if $\nu > \bar{\nu}$. In this case, the optimal policy is given by equation (16).*

Proof. The first-order condition for the policy maker's choice of interest rate can be written as

$$\alpha\lambda_2 \left[\nu u'\left(q_1\left(1+i\right)\right) - w'\left(q_1\left(1+i\right)\right)\right] q_1'\left(1+i\right) \leq 0,$$

with equality if $i > 0$. The proof of Proposition 1 establishes that q_1 is strictly increasing in $1+i$. Therefore, the optimal choice of interest rate has $i > 0$ if and only if the expression in square brackets is positive when evaluated at $i = 0$, that is,

$$\nu u'\left(q_1^N\right) - w'\left(q_1^N\right) > 0 \quad \text{or} \quad \nu > \frac{w'\left(q_1^N\right)}{u'\left(q_1^N\right)} \equiv \bar{\nu}.$$

Because q_1^N satisfies

$$\frac{\mu}{\beta} = \alpha\frac{u'\left(q_1^N\right)}{w'\left(q_1^N\right)} + 1 - \alpha,$$

we have

$$\bar{\nu} = \frac{\alpha\beta}{\mu - (1-\alpha)\beta}.$$

By varying $1+i$, the policy maker can implement any quantity of DM trade between q_1^N and q^* in type 1 meetings. The optimal choice has the property that q_1 satisfies

$$\frac{w'\left(q_1\right)}{u'\left(q_1\right)} = \nu \leq 1$$

The equilibrium value of q_1 when $i > 0$ satisfies the first-order condition

$$\frac{\mu}{\beta\left(1+i\right)} = \alpha\frac{u'\left(q_1\right)}{w'\left(q_1\right)} + 1 - \alpha.$$

Combining these equations yields the optimal policy in equation (16). □

Proposition 4. *With a deposit-like digital currency satisfying condition (17), the unique equilibrium allocation satisfies $e_2^D + d^D > d^N > d^D$, $\left(r^D, \hat{\gamma}^D, q_2^D\right) \gg \left(r^N, \hat{\gamma}^N, q_2^N\right)$, and $q_1^D = q_1^N$.*

Proof. When $1+i$ is set so that equation (17) holds, the equilibrium interest rate and the investment cutoff rise to

$$1 + r^D = \frac{1+i}{\mu} > 1 + r^N \quad \text{and} \quad \hat{\gamma}^D = \frac{1+i}{\mu\theta} > \hat{\gamma}^N,$$

respectively. The new quantity of deposits is determined by the supply function at the investment cutoff point:

$$d^D = \frac{\eta}{\lambda_2}\left(\bar{\gamma} - \frac{1+i}{\theta\mu}\right).$$

The quantity of digital currency held by buyers heading into a type 2 meeting is

$$e_2^D = \frac{\mu}{1+i} L^{-1} \left(\frac{\mu}{\beta (1+i)} \right) - \frac{\eta}{\lambda_2} \left(\bar{\gamma} - \frac{1+i}{\theta \mu} \right).$$

Note that the resulting value of e_2^D is positive if and only if $1+i > \mu \left(1+r^N\right)$, and e_2^D is a strictly increasing function for $1+i \in \left(\mu \left(1+r^N\right), \mu/\beta\right)$. It is also easy to show that d^D is strictly decreasing in $1+i$ in this region and that the sum $e_2^D + d^D$ is strictly increasing.

The quantity traded in type 2 meetings satisfies

$$\frac{\mu}{\beta (1+i)} = \alpha \frac{u'\left(q_2^D\right)}{w'\left(q_2^D\right)} + 1 - \alpha.$$

Given our assumptions on preferences, we have $q_2^D > q^N$ when $1+i > \mu \left(1+r^N\right)$. □

Proposition 5. *There exists $\bar{\eta} > 0$ such that $\eta < \bar{\eta}$ implies a deposit-like digital currency is desirable. The optimal policy satisfies $1+i \in \left[\mu \left(1+r^N\right), \mu/\beta\right)$ if $\theta < 1$ and $1+i = \mu/\beta$ if $\theta = 1$.*

Proof. The objective function in equation (20) need not be concave in $1+i$, but it is continuous on the closed interval $1+i \in \left[\mu \left(1+r^N\right), \mu/\beta\right]$ and, therefore, an optimal policy exists. To characterize this policy, it is useful to set up an *auxiliary problem* in which the policy maker directly chooses a real interest rate $1+r \in (0, \frac{1}{\beta}]$ to maximize

$$\hat{W}(1+r) \equiv \eta \int_{\frac{1+r}{\theta}}^{\bar{\gamma}} (\beta \gamma - 1)\, d\gamma + \alpha \lambda_2 \left[u\left(\hat{q}_2(1+r)\right) - w\left(\hat{q}_2(1+r)\right) \right],$$

where

$$\hat{q}_2(1+r) = w^{-1} \left(\beta L^{-1} \left(\frac{1}{\beta (1+r)} \right) \right).$$

Unlike with a CBDC, which only allows the policy maker to increase the equilibrium deposit rate, this auxiliary problem allows the policy maker to either increase or decrease $1+r$. The Inada conditions on u imply that $\hat{W}$ is strictly increasing when $1+r$ is sufficiently close to zero and, therefore, the auxiliary problem has a solution. Let $1+\hat{r}$ denote this solution. (If there are multiple solutions, let $1+\hat{r}$ denote the smallest one.)

If $1+\hat{r} > 1+r^N$, then, by definition, we must have $\hat{W}(1+r) > \hat{W}\left(1+r^N\right)$. Moreover, the nominal interest rate $1+i = \mu(1+\hat{r})$ is contained in the policy maker's choice set for the original problem and, therefore, introducing a digital currency that bears this interest rate raises welfare. To establish the first part of the proposition, therefore, it suffices to show that $1+\hat{r} > 1+r^N$ holds when η is sufficiently small.

The slope of the auxiliary objective $\hat{W}$ is given by

$$\frac{d\hat{W}}{d(1+r)} = -\frac{\eta}{\theta} \left[\frac{\beta(1+r)}{\theta} - 1 \right] + \frac{\lambda_2}{-L'\left(L^{-1}\left(\frac{1}{\beta(1+r)}\right)\right)} \frac{1 - \beta(1+r)}{\beta(1+r)^3}$$

The second term in this expression is positive for all values of $1+r<\beta^{-1}$. The first term is also positive when $1+r<\theta/\beta$ and, therefore, the solution to the auxiliary problem must satisfy

$$1+\hat{r}>\frac{\theta}{\beta}$$

for all values of η. As established in Proposition 1, $1+r^N$ is a strictly increasing function of η. Moreover, as η approaches the lower bound in Assumption 1, $1+r^N$ approaches zero. It follows that there exists $\bar{\eta}>0$ such that $1+r^N<1+\hat{r}$ holds for all $\eta<\bar{\eta}$, which establishes the first part of the proposition.

For the second part of the proposition, first assume $\theta<1$ and evaluate the derivative in equation (22) at $1+i=\mu/\beta$. At this interest rate, there is no liquidity premium, which implies that type 2 buyers will be satiated in real balances and the quantity produced in type 2 DM meetings will be q^*. The second term in the derivative is thus zero. Because $\theta<1$, the first term in the derivative is negative. It follows that the solution to the optimal policy problem must be lower, with $1+i<\mu/\beta$.

Finally, when $\theta=1$, the derivative in equation (22) is strictly positive for all $1+i<\frac{\mu}{\beta}$ and, therefore, the optimal policy is $1+i=\frac{\mu}{\beta}$. □

Proposition 6. *The optimal policy under a universal CBDC implements the same allocation as under two restricted-use CBDCs if and only if at least one of the following conditions holds:*

$$\begin{aligned}
&(i)\ 1+i^C=1 \quad \text{and} \quad 1+i^D=\mu\left(1+r^N\right),\\
&(ii)\ 1+i^C=1+i^D,\\
&(iii)\ 1+i^C\leq 1+i^D=\mu\left(1+r^N\right), \quad \text{or}\\
&(iv)\ 1+i^C=1\geq 1+i^D.
\end{aligned}$$

Proof. Define $1+r^C\equiv\left(1+i^C\right)/\mu$, $1+r^D\equiv\left(1+i^D\right)/\mu$, and $1+r^U\equiv\left(1+i^U\right)/\mu$ as the optimal real interest rate for the cash-like, deposit-like, and universal CBDCs, respectively. Note that the aforementioned set of conditions can be written in terms of the real interest rate as

$$\begin{aligned}
(i)\ 1+r^C &= \frac{1}{\mu} \quad \text{and} \quad 1+r^D=1+r^N,\\
(ii)\ 1+r^C &= 1+r^D,\\
(iii)\ 1+r^C &\leq 1+r^D=1+r^N, \text{ or}\\
(iv)\ 1+r^C &= \frac{1}{\mu}\geq 1+r^D.
\end{aligned}$$

The equilibrium allocation $(q_1,q_2,\hat{\gamma})$ under a universal CBDC will be the same as with two restricted-use CBDCs if and only if each type of buyer faces the same rate of return on spendable assets under both regimes. For type 1 buyers, this requirement can be written as

$$\text{If } 1+r^C>\frac{1}{\mu}, \text{ then } 1+r^U=1+r^C; \text{ otherwise, } 1+r^U\leq\frac{1}{\mu}. \tag{A2}$$

In other words, if a cash-like CBDC is desirable, the interest rate on a universal CBDC must equal the optimal cash-like rate. If a cash-like CBDC is not desirable, the interest rate on the universal currency must be low enough that it does not change the quantity of real balances held by type 1 buyers. The requirement for type 2 buyers is

$$\text{If } 1+r^D > 1+r^N, \text{ then } 1+r^U = 1+r^D; \text{ otherwise, } 1+r^U \leq 1+r^N. \tag{A3}$$

The logic here is similar. If a deposit-like CBDC is desirable, the interest rate on a universal CBDC must equal the optimal deposit-like rate. If it is not desirable, the interest rate on the universal currency must be low enough that no type 2 buyer chooses to hold it.

It is straightforward to show that each of the four conditions in the proposition is sufficient to guarantee that the requirements (A2) and (A3) are satisfied. Under condition (*i*), the optimal policy is $1+r^U = \min\left\{1/\mu, 1+r^N\right\}$. Under condition (*ii*), it is $1+r^U = 1+r^C = 1+r^D$. Under condition (*iii*), the optimal policy sets $1+r^U = 1+r^C$ to satisfy requirement (A2); the inequality in the condition then guarantees that (A3) is satisfied as well. Finally, under condition (*iv*), the optimal policy sets $1+r^U = 1+r^D$ to satisfy (A3) and the inequality in the condition guarantees that (A2) is also satisfied.

The less obvious part of the proposition is that it is also necessary for at least one of these conditions to hold if a universal CBDC is to implement the same allocation as with two restricted-use CBDCs. We establish this part of the result by showing that if conditions $(i)-(iii)$ are **not** satisfied, then requirements (A2) and (A3) imply that condition (*iv*) is necessarily satisfied.

We begin with conditions (*i*) and (*ii*). If condition (*i*) is not satisfied, then we either have $1+r^C > 1/\mu$ or $1+r^D > 1+r^N$, or both. In other words, at least one type of restricted-use CBDC is desirable. If condition (*ii*) is not satisfied, the desired interest rates in the two types of meetings are different. Requirements (A2) and (A3) then imply that either $1+r^C = 1/\mu$ or $1+r^D = 1+r^N$ must hold. In other words, if neither (*i*) nor (*ii*) is satisfied and a universal CBDC can implement the same allocation as two restricted-use CBDCs, it must be the case that one restricted-use CBDC is desirable and the other is not.

Now suppose that, in addition, condition (*iii*) is not satisfied, meaning either $1+r^C > 1+r^D$ or $1+r^D > 1+r^N$. Suppose first that $1+r^C > 1+r^D$ held. Requirements (A2) and (A3) imply

$$1+r^U \leq \min\left\{1+r^C, 1+r^D\right\} \tag{A4}$$

and, therefore, we would have $1+r^U < 1+r^C$. Requirement (A2) would then imply $1+r^C = 1/\mu$ must hold. Given that we have supposed $1+r^C > 1+r^D$, it follows that condition (*iv*) is satisfied. If we instead suppose $1+r^D > 1+r^N$ held, then (A3) would require $1+r^U = 1+r^D$ and the fact that only one restricted-use CBDC is desirable would imply $1+r^C = 1/\mu$. Combining these results with equation (A4) would imply $1+r^D \leq 1/\mu$ and condition (*iv*) is again satisfied, as desired. □

Proposition 7. *A universal digital currency is desirable if any of the following sets of conditions holds:*

$$(i)\ \nu > \frac{\alpha\beta}{\mu-(1-\alpha)\beta} \quad \text{and} \quad 1+r^N > \frac{1}{\mu};$$
$$(ii)\ \nu > \frac{\alpha\beta}{\mu-(1-\alpha)\beta} \quad \text{and} \quad 1+r^N < \frac{\theta}{\beta}; \quad \text{or}$$
$$(iii)\ 1+r^N < \frac{1}{\mu} \quad \text{and} \quad 1+r^N < \frac{\theta}{\beta}.$$

Proof. To establish that a universal CBDC is desirable, we must show that there exists an interest rate $1+i$ such that welfare $W^U(1+i)$ from equation (23) is strictly higher than welfare with no digital currency, W^N. We address each of the three cases in turn.

(i) Because $\nu > \bar{\nu}$, a CBDC that pays a positive but sufficiently small interest rate will increase the middle term on the right-hand side of equation (23), which corresponds to the surplus from type 1 DM meetings. Moreover, $\mu\left(1+r^N\right) > 1$ implies that if the net CBDC interest rate is set sufficiently close to zero, it will not be held by type 2 buyers and will not affect CM investment. It follows that there exists $\varepsilon > 0$ such that a universal CBDC with interest rate $1+i \in (1, 1+\varepsilon)$ raises welfare.

(ii) As in case (i), a CBDC with a sufficiently small positive interest rate will increase the surplus from type 1 DM meetings. Separately, $1+r^N < \theta/\beta$ implies that there is overinvestment in the equilibrium with no digital currency and, therefore, setting the CBDC interest rate slightly above $\mu\left(1+r^N\right)$ would increase both the first and third terms on the right-hand side of equation (23). By choosing the CBDC interest rate to be slightly above the *smaller* of these two values, the policy maker can ensure that some welfare terms increase while no others decrease. In other words, there exists $\varepsilon > 0$ such that a universal CBDC with interest rate

$$1+i \in \left(min\left\{1, \mu\left(1+r^N\right)\right\}, min\left\{1, \mu\left(1+r^N\right)\right\}+\varepsilon\right)$$

raises welfare.

(iii) In this case, there is again overinvestment in the equilibrium with no digital currency, so setting $1+i$ slightly larger than $\mu\left(1+r^N\right)$ will increase the first and third terms on the right-hand side of equation (23). In addition, $\mu\left(1+r^N\right) < 1$ implies that setting the CBDC interest rate sufficiently close to $\mu\left(1+r^N\right)$ ensures that it will not be used in type 1 DM meetings. If follows that there exists $\varepsilon > 0$ such that a universal CBDC with interest rate $1+i \in \left(1+r^N, 1+r^N+\varepsilon\right)$ raises welfare. □

Corollary 3. *If* $\nu > \frac{\alpha\beta}{\mu-(1-\alpha)\beta}$*, the optimal interest rate on a universal CBDC satisfies* $1+i \geq \frac{\theta\mu}{\beta}$.

Proof. The proof follows similar reasoning to case (ii) in Proposition 7. Assume that $(1+i)/\mu < \theta/\beta$ holds at the optimum. Because this condition implies overinvestment, we can find an $\varepsilon > 0$ with $(1+i)/\mu < (1+i+\varepsilon)/\mu < \theta/\beta$ such that the interest rate $1+i+\varepsilon$ results in a higher value for the welfare function, W^U. But this contradicts the fact that $(1+i)/\mu$ is a solution to the welfare maximization problem. □