Solutions to Problem Set #1

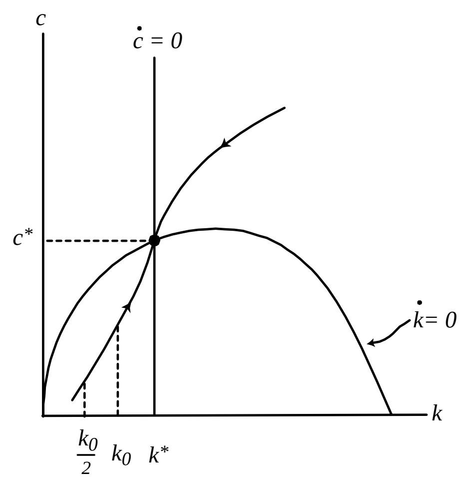
Economic Growth
Spring 2005

Professor Todd Keister
keister@itam.mx

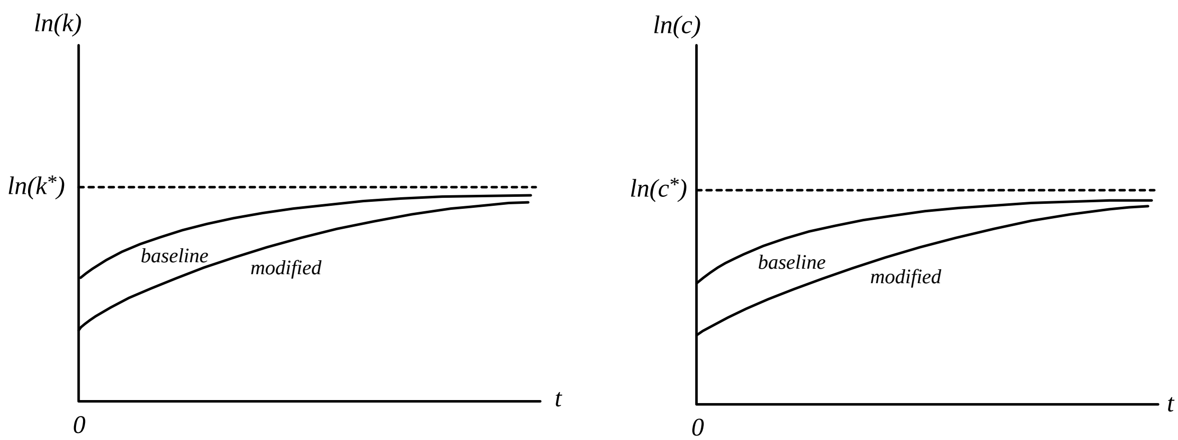
1) Consider the optimal growth problem from class, with $u(c) = \ln(c)$ and $n = 0$, and compare the following two situations. In the baseline case, the island has $k_0 < k^*$ trees when Crusoe arrives. In the modified case, the island only has half as many trees $\left(\frac{1}{2}k_0\right)$ when Crusoe arrives. Everything else is the same in the two cases.

a) Draw the time paths of k and c for these two cases. How do they compare?

In both cases, the economy will converge to the same steady state (k^*, c^*). They will follow the same stable arm of the saddle point, but from different starting locations.



As the phase diagram indicates, in the baseline case (where there are more trees) the initial level of consumption will be higher. The time paths must therefore look like



Both the number of trees and the amount of consumption are always higher in the baseline case, but in the long run the modified case "catches up" as both economies converge to the steady state.

b) Suppose that we look at the growth rate of consumption (that is, $\dot{c}/c$) in each of these two cases. At time $t = 0$, in which case will the growth rate be higher? Why? (Give some intuition for your answer).

From the differential equation for c we have

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho.$$

Since f is concave, the right-hand side is decreasing in $k(t)$. The economy that starts with fewer trees will therefore experience faster growth. This is exactly what we saw in the time paths above: the economy that starts out with fewer trees grows faster and in the long run will catch up to the initially "richer" economy. The reason for this is the decreasing returns to scale in the intensive ("per capita") harvesting technology. When there are fewer trees, each person spends more time on each tree and therefore gets more coconuts per tree. In other words, the trees are more productive, which leads to a faster growth rate.

2) Now consider the following changes in the model. First, instead of the general harvesting function $Y(t) = F(K(t), N(t))$, use the Cobb-Douglas function $Y(t) = K(t)^{\alpha} N(t)^{1-\alpha}$. Second, instead of the utility function $u(c) = \ln(c)$, the Crusoe household's preferences are represented by

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta},$$

with $\theta > 0$. Note that when $\theta = 1$, this function is not well defined (it gives "zero divided by zero" for every value of c). However, as $\theta \to 1$, the function converges to $\ln(c)$. Finally, assume $n > 0$.

a) Write down the optimal growth problem, the Hamiltonian function for this problem, and the 3 first-order conditions. Also write down the transversality condition.

The first step is to derive the intensive production function.

$$\begin{aligned} y(t) &= \frac{K(t)^{\alpha} N(t)^{1-\alpha}}{N(t)} \\ &= \frac{K(t)^{\alpha}}{N(t)^{\alpha}} = k(t)^{\alpha}. \end{aligned}$$

Then we can write the optimal growth problem as

$$\max \int_0^{\infty} \frac{c(t)^{(1-\theta)} - 1}{1-\theta} e^{-(\rho-n)t} dt$$

subject to

$$\begin{aligned} \dot{k}(t) &= k(t)^{\alpha} - c(t) - (\delta + n) k(t) \\ k(0) &= k_0 \\ k(t), c(t) &\geq 0 \quad \text{for all } t. \end{aligned}$$

The Hamiltonian function is given by

$$H(t) = \frac{c(t)^{(1-\theta)} - 1}{1-\theta} e^{-(\rho-n)t} + \mu(t) \left[k(t)^{\alpha} - c(t) - (\delta + n)\, k(t) \right],$$

and the FOC are given by

$$\begin{aligned}
&\text{(a) } \frac{\partial H}{\partial c} &=& \; 0 &\quad \Rightarrow \quad & c(t)^{-\theta} e^{-(\rho-n)t} = \mu(t) \\
&\text{(b) } \frac{\partial H}{\partial k} &=& \; -\dot{\mu} &\quad \Rightarrow \quad & \mu(t) \left[\alpha k(t)^{\alpha-1} - \delta - n \right] = -\dot{\mu}(t) \\
&\text{(c) } \frac{\partial H}{\partial \mu} &=& \; \dot{k} &\quad \Rightarrow \quad & \dot{k}(t) = k(t)^{\alpha} - c(t) - (\delta + n)\, k(t).
\end{aligned}$$

The transversality condition is

$$\lim_{t \to \infty} \mu(t) k(t) = 0.$$

b) Solve the first-order conditions to get a system of two differential equations in the variables (k, c) *. How do these differ from the equations we saw in class?*

This follows what we did in class very closely. Taking the log of both sides of (a) gives

$$-\theta \ln [c(t)] - (\rho - n)\, t = \ln [\mu(t)] \, .$$

Next differentiate with respect to time to get

$$-\theta \frac{\dot{c}(t)}{c(t)} - (\rho - n) = \frac{\dot{\mu}(t)}{\mu(t)}.$$

Substituting this into (b) gives us the differential equation for $c(t)$. Equation (c) gives us directly the differential equation for k. These equations are

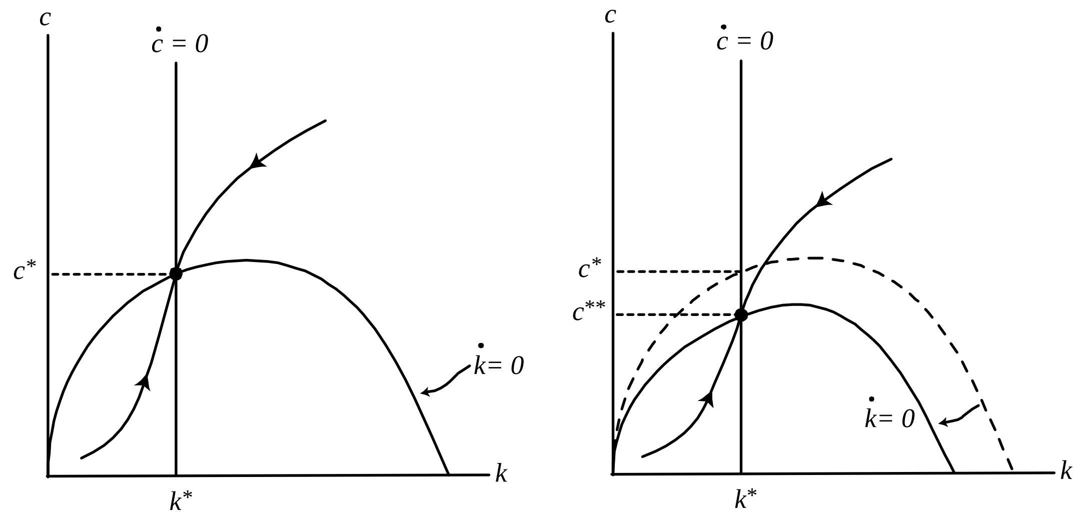
$$\begin{aligned}
\dot{c}(t) &= \frac{1}{\theta} \left[\alpha k(t)^{\alpha-1} - \delta - \rho \right] c(t) \\
\dot{k}(t) &= k(t)^{\alpha} - c(t) - (\delta + n)\, k(t)
\end{aligned}$$

There are a couple of important things to notice here. First, n does not appear in the $\dot{c}$ equation. It appeared in the first-order conditions (a) and (b), but it cancelled out when we combined them. Why? [Think about this. The variable n has two effects on this equation, and they are exactly cancelling out. What are these two effects?] Second, the constant θ is appears in the first equation (here we can see that $\theta = 1$ corresponds to the case of log utility). Finally, the functions f and f' have been replaced by specific functions involving α.

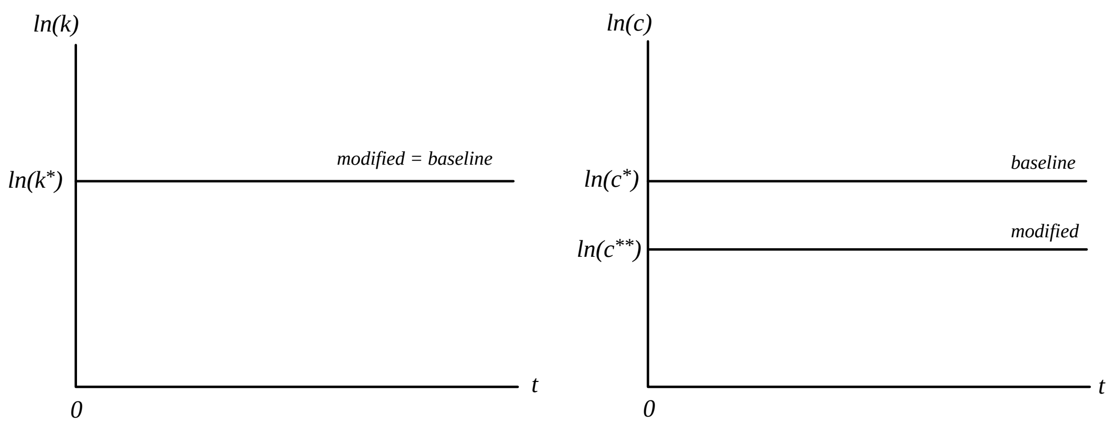
c) Do the following comparative dynamics exercise: $n' > n$. *Proceed exactly as we did in class. Draw the phase diagram for the baseline case, and suppose that* k_0 *is equal to* k^* *for this case. Then draw the modified phase diagram, indicating what has changed with the higher value of* n.

Draw the modified time paths of k and c, indicating how they compare with the baseline time paths. What effect does the increase in n have? Why? (Give an intuitive answer.)

The isocline for c does not depend on n and hence does not change. The isocline for k shifts downward, because in the resource constraint an increase in n is just like an increase in the death rate of trees. Therefore, the phase diagram looks like:



The level of c must adjust to put the economy somewhere on the stable arm leading to the modified steady state. For the diagram we drew above, this means that the modified economy must start in its steady state. The level of trees per person is the same in both cases. The level of consumption per person is therefore constant in both cases, but at different levels.



Faster population growth means more planting is required to keep the level of trees per person constant. The higher level of planting implies that fewer coconuts are consumed in the steady state.

Why is the steady state level of trees per person the same in both cases? Think again about the answer in part (b). There are two effects which exactly cancel each other out. What are these effects?