Solutions to Problem Set #2

Economic Growth — Professor Todd Keister
Spring 2005 — keister@itam.mx

1) *Consider the optimal growth problem with the CIES utility function*

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$$

and the Cobb-Douglas production function $Y(t) = BK(t)^{\alpha} N(t)^{1-\alpha}$ *, where* $B > 0$ *is a constant. Assume* $n > 0$.

a) Write down the complete optimal growth problem. Solve this problem using the Hamiltonian method and derive two differential equations in the variables (c, k) *.*

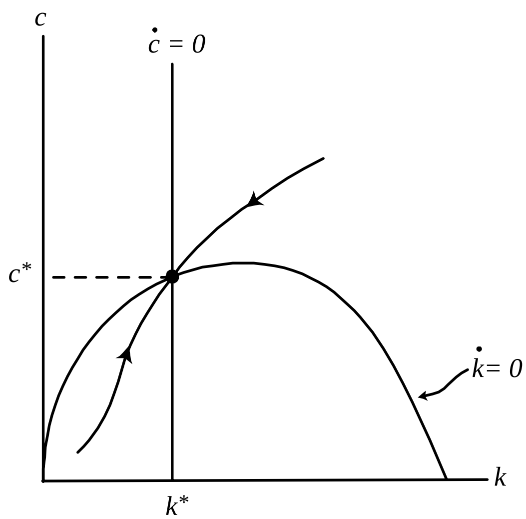
The steps are identical to those in Question 2 on Problem Set #1. Therefore I will skip directly to the differential equations:

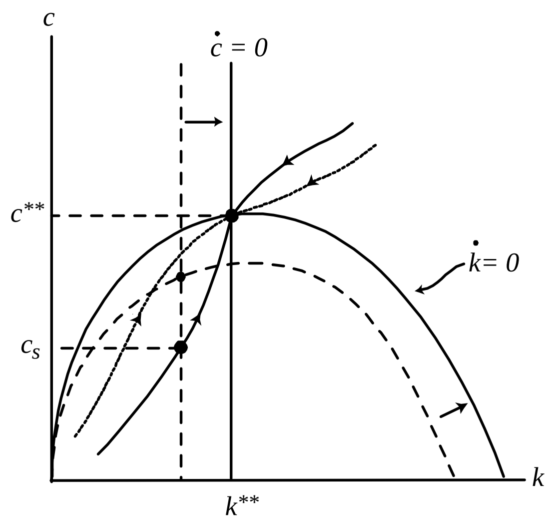
$$\begin{aligned} \dot{c}(t) &= \frac{1}{\theta}\left[\alpha B k(t)^{\alpha-1} - \delta - \rho\right] c(t) \\ \dot{k}(t) &= Bk(t)^{\alpha} - c(t) - (\delta + n)\, k(t) \end{aligned}$$

The constant B appears in the marginal product of capital in the first equation, and in the production function in the resource constraint.

b) Do the following comparative dynamics exercise: $B' > B$, *following the usual steps. ... Pay special attention to the value of* c *at* $t = 0$; *is it higher or lower in the modified case? Why? (Give an intuitive answer.)*

In the modified case, the isocline for c shifts to the right. The isocline for k is where $c = Bk^{\alpha} - (\delta + n)\, k$ holds. When B is higher, this isocline rotates upward. The phase diagrams are drawn below.



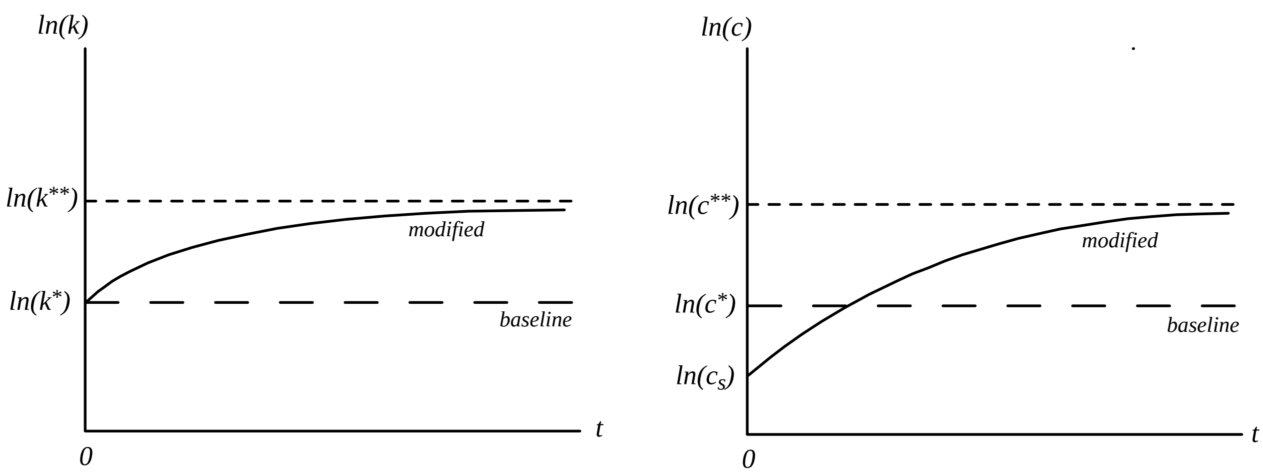


From this diagram we can see that the stable arm of the modified steady state could pass either above or below the baseline steady state. There must be two effects that are pushing $c(0)$ in opposite direction.

Substitution effect: The increase in B makes consumption in the future less expensive relative to consumption today (because the production technology is better), so consumption today (at $t = 0$) should tend to decrease. This corresponds to the rightward shift in the isocline for c.

Income effect: The increase in B implies that Crusoe is richer, so consumption should tend to increase at all points in time, including $t = 0$. This corresponds to the upward rotation in the isocline for k.

Whether c_S for the modified case is larger or smaller than c^* depends on which of these two effects dominates. I will assume that the substitution effect dominates, so that the appropriate stable arm for the modified case is the solid one in the figure above. The time paths of k and c are then:



2*) Consider the Ramsey model of an economy in competitive equilibrium. There is a representative household and a representative firm. Assume there is no population growth* $(n = 0)$ *. The household's utility function is*

$$\int_0^\infty \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt,$$

and the firm has a constant-returns-to-scale production function $Y(t) = F[K(t), L(t)]$ *. Everything is the same as what we saw in class, with the following exception. In class, we assumed that one unit of output that is not consumed becomes one unit of capital. Now, one unit of output that is not consumed becomes* $\sigma < 1$ *units of capital. We can think of the parameter* σ *as measuring the efficiency of the financial sector. One of the primary roles of the financial sector is to put the savings of households to productive use. When* σ *is low, it takes a lot of saving to create one new productive machine. In this sense, we can say that the financial sector is inefficient.*

Find the competitive equilibrium of this economy, using the following steps.

a) Write down representative household's maximization problem and derive the 4 equations that characterize the solution.

There is no change in the household's problem. The goods that are saved by the household are still deposited in the bank and still earn some interest rate r. Recall that $a(t)$ is measured in units of output (not capital). Hence the answer here is exactly the same as in class:

$$\begin{aligned} & \max_{\{c(t)\}} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt \\ & s.t. \\ \dot{a}(t) &= w(t) + r(t) a(t) - c(t) \\ a(0) &= a_0 \\ a(t) &\geq -B \text{ for all } t. \\ c(t) &\geq 0 \text{ for all } t. \end{aligned}$$

and

$$\dot{c}(t) = \frac{1}{\theta}[r(t)-\rho] c(t) \quad (1)$$

$$\dot{a}(t) = w(t) + r(t) a(t) - c(t) \quad (2)$$

$$\lim_{t\to\infty} \mu(t) a(t) = 0 \quad (3)$$

$$c(t) \geq -B \text{ for all } t \quad (4)$$

b) Write down firm's maximization problem and the first-order conditions for this problem. Translate these conditions into intensive form.

There is no change to the firm's problem, either. We still have

$$\max\ F(K(t), L(t)) - w(t) L(t) - R(t) K(t)$$

subject to

$$K(t), L(t) \geq 0$$

Following exactly the same steps as in class, we arrive at

$$f'(k(t)) = R(t) \quad (5)$$

$$f(k(t)) - k(t) f'(k(t)) = w(t) \quad (6)$$

c) What are the equilibrium conditions for this economy?

The labor-market clearing equation is unchanged:

$$N(t) = L(t). \quad (7)$$

Now, finally, we need to think about how the activities of banks are affected by σ. First let's examine the bank's balance sheet. For each unit of goods that the bank has received as deposits, it has created σ units of capital. The total amount of deposits it has received is $A(t)$, so we must have

$$K(t) = \sigma A(t)$$

or, in intensive form,

$$k(t) = \sigma a(t). \tag{8}$$

Next, think about the bank's profit. (This is the most difficult part.) Measured in goods, total profit at time t is

$$\left[R(t) - \frac{\delta}{\sigma}\right] k(t) - r(t)\, a(t).$$

Why is δ on the left-hand side divided by σ? Because we are measuring everything in units of goods here. For example, $R(t)$ is the number of goods that a firm pays to the bank when it rents a unit of capital. The number $\delta k(t)$ is measured in machinery: it is the number of machines that need to be replaced. We need to ask what the value of these machines is in terms of goods: How many goods are required to replace $\delta k(t)$ worn-out machines? The answer is $\frac{\delta}{\sigma} k(t)$. The zero-profit condition for the bank is then

$$\sigma R(t) - \delta = r(t). \tag{9}$$

d) Combine your answers to parts (a) - (c) and derive a pair of differential equations for the variables c and k.

Combining the nine equations above together (the same way we did in class) leads to the equations

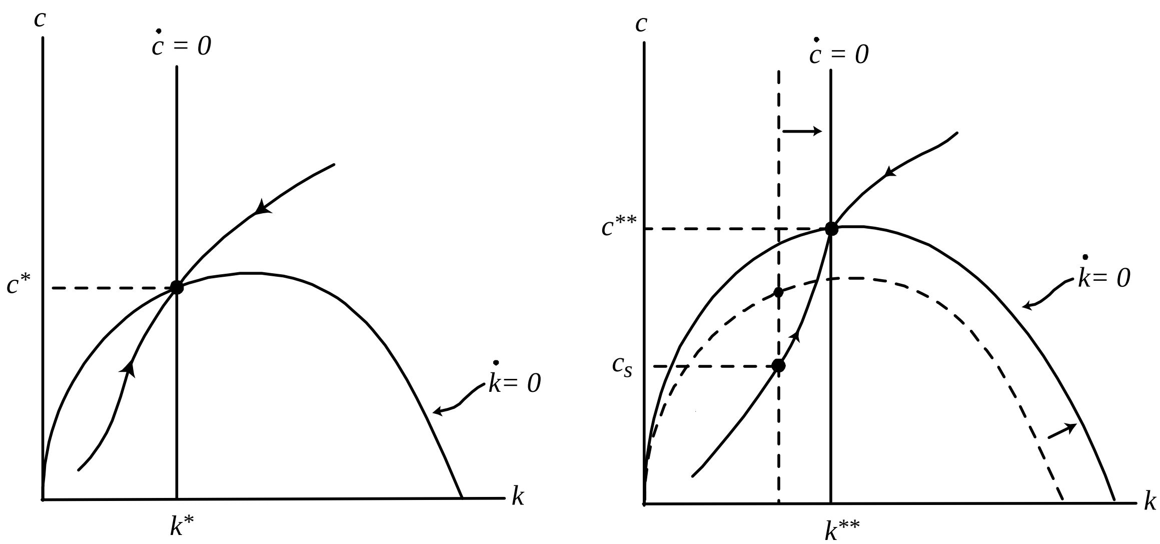
$$\begin{aligned}
\dot{c}(t) &= \frac{1}{\theta}\left[\sigma f'(k(t)) - \delta - \rho\right] c(t) \\
\dot{k}(t) &= \sigma\left[f(k(t)) - c(t)\right] - \delta k(t)
\end{aligned}$$

e) Do the following comparative dynamics exercise: $\sigma' > \sigma$. As usual, the baseline economy starts in its steady state at time $t = 0$. The modified economy starts at time $t = 0$ with the same amount of capital as the baseline economy. Draw (i) the phase diagram for both cases, indicating what is different, and (ii) the time paths of c and k for both cases. If necessary, assume that the substitution effect dominates the income effect.

The isoclines for the phase diagram are given by

$$\begin{aligned}
\dot{c} &= 0 : f'(k) = \frac{\delta + \rho}{\sigma} \\
\dot{k} &= 0 : c = f(k(t)) - \frac{\delta}{\sigma} k(t).
\end{aligned}$$

These expressions show that the effect of a higher value of σ will be similar to having a lower value of δ. That is, having a more efficient financial sector will be similar to having a better technology for maintaining machines. For the modified case, the isocline for c shifts to the right (to a higher value k^{**}) and the isocline for k rotates up.



In drawing the stable arm of the saddle point, we see that the income and substitution effects point in opposite directions. When σ is higher, the household is richer and the income effect points toward a higher value of $c(0)$. However, saving is also more productive and hence the substitution effect points toward a lower value of $c(0)$. The figure above is drawn with c_S smaller than c^*, indicating that the substitution effect is dominant. The corresponding time paths are