Problem Set #3

Economic Growth
Spring 2005

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1) Consider the following problem

$$\max_{\{c(t)\}} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-(\rho-n)t} dt$$

subject to

$$\begin{aligned}
\dot{k}(t) &= F(k(t), A(t)) - c(t) - (\delta+n)k(t), \\
k(0) &= k_0, \\
k(t) &\geq 0 \text{ for all } t, \text{ and} \\
c(t) &\geq 0 \text{ for all } t.
\end{aligned}$$

This is the optimal growth problem (also called the Pareto problem) when there is exogenous technological progress. Notice that it is the same as the problem you worked with in Problem Set #1, except that $A(t)$ appears in the production function. Assume, as in class, that the level of productivity A grows at the constant rate $g > 0$.

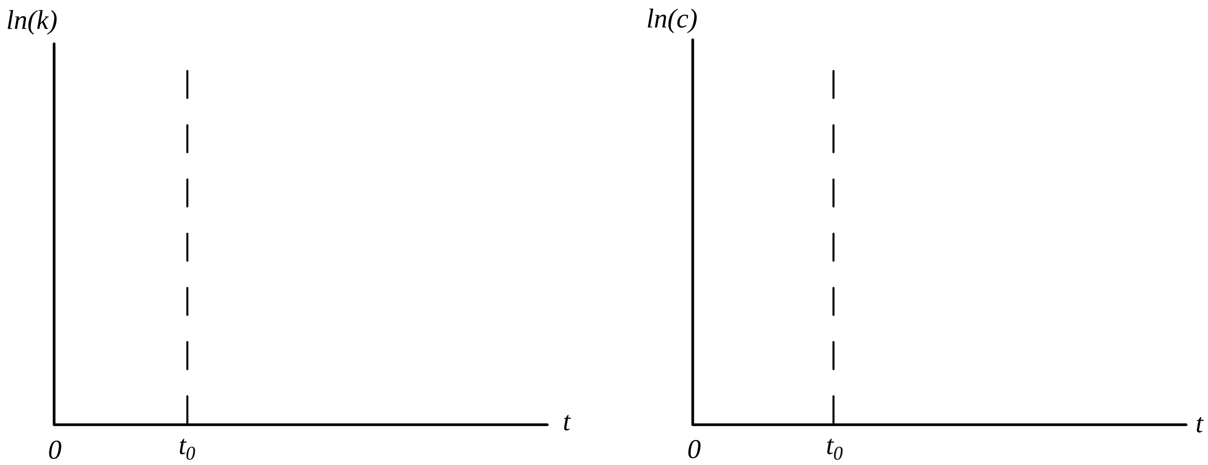
a) Write the Hamiltonian function for this problem, and the 3 first-order conditions. Also write down the transversality condition.

b) Solve the first-order conditions to get a system of two differential equations in the variables (k, c). (These equations should have the variable A in them.)

c) Define the variables $\widehat{k}$ and $\widehat{c}$ as we did in class, so that they represent capital per effective worker and consumption per effective worker, respectively. Derive a pair of differential equations in the variables $\left(\widehat{k}, \widehat{c}\right)$.

d) In class we derived the equations for $\widehat{k}$ and $\widehat{c}$ that characterize the *equilibrium* of the Ramsey model with technological progress. How does your answer in part (c) compare to the equations we derived in class? Why?

2) Use the model in question (1) to analyze the following situation. Suppose an economy is initially on its balanced growth path, with $\left(\widehat{k}, \widehat{c}\right) = \left(\widehat{k}^*, \widehat{c}^*\right)$. Then, at some point in time $t_0 > 0$, suddenly and unexpectedly, the value of k falls by one-half. (Imagine, for example, that there is a war or a natural disaster and half of the factories are destroyed.) Draw the time paths of k and c from time 0 onwards. Note that this exercise is different from the comparative dynamics exercises we have done before in that there is only one economy; the baseline case represents the economy before t_0 and the modified case represents the same economy after t_0. Your graphs should look like the picture below, so that they show (i) what was happening before time t_0, when the economy was on the balanced growth path, (ii) the sudden decrease in capital and the corresponding change in consumption at t_0, and (iii) what happens after the economy has suffered this loss.



[Note: I have said in class that state variables, like k, cannot make sudden changes. This is generally true; the stock of machinery moves gradually in response to investment decisions, and therefore k will be a continuous function of time under "normal" circumstances. However, there can be exceptional cases like the one described above: If a lot of machines are suddenly blown up by bombs, then there are suddenly fewer machines, and the function $k(t)$ jumps down.]

3) Again using the problem in question (1), do the following comparative dynamics exercise: $\delta' > \delta$. Draw the time paths for the variables k and c for the baseline and the modified cases.

This exercise is like the comparative dynamics exercises we have done in class and in earlier homework problems. Draw the phase diagram for the baseline case, and suppose that $\widehat{k}_0$ is equal to $\widehat{k}^*$ for this case. Then draw the modified phase diagram, indicating what has changed with the higher value of δ. Draw the modified time paths of k and c, indicating how they compare with the baseline time paths. (I would suggest first drawing the time paths of $\widehat{k}$ and $\widehat{c}$.) If necessary, assume that the substitution effect dominates the income effect. Give some intuition for your results.