Solutions to Problem Set #3

Economic Growth | Professor Todd Keister
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1) Consider the following problem

$$\max_{\{c(t)\}} \int_0^\infty \frac{c(t)^{1-\theta}-1}{1-\theta} e^{-(\rho-n)t} dt$$

subject to

$$\begin{aligned} \dot{k}(t) &= F(k(t), A(t)) - c(t) - (\delta+n)k(t), \\ k(0) &= k_0, \text{ and} \\ k(t), c(t) &\geq 0 \text{ for all } t. \end{aligned}$$

This is the optimal growth problem (also called the Pareto problem) when there is exogenous technological progress. Assume that the level of productivity A grows at the constant rate $g > 0$.

a) Write the Hamiltonian function for this problem, and the 3 first-order conditions. Also write down the transversality condition.

The Hamiltonian function is given by

$$H(t) = \frac{c(t)^{(1-\theta)}-1}{1-\theta} e^{-(\rho-n)t} + \mu(t)\left[F(k(t), A(t)) - c(t) - (\delta+n)k(t)\right].$$

The FOC are given by

$$\begin{aligned} \text{(a)} \frac{\partial H}{\partial c} &= 0 \quad \Rightarrow \quad c(t)^{-\theta} e^{-(\rho-n)t} = \mu(t) \\ \text{(b)} \frac{\partial H}{\partial k} &= -\dot{\mu} \quad \Rightarrow \quad \mu(t)\left[F_K(k(t), A(t)) - (\delta+n)\right] = -\dot{\mu}(t) \\ \text{(c)} \frac{\partial H}{\partial \mu} &= \dot{k} \quad \Rightarrow \quad \dot{k}(t) = F(k(t), A(t)) - c(t) - (\delta+n)k(t) \end{aligned}$$

The transversality condition is

$$\lim_{t\to\infty} \mu(t)k(t) = 0.$$

b) Solve the first-order conditions to get a system of two differential equations in the variables (k, c). (These equations should have the variable A in them.)

Following the usual steps, we get

$$\begin{aligned} \dot{c}(t) &= \frac{1}{\theta}\left[F_K(k(t), A(t)) - \delta - \rho\right] c(t) \\ \dot{k}(t) &= F(k(t), A(t)) - c(t) - (\delta+n)k(t) \end{aligned}$$

c) Define the variables $\widehat{k}$ and $\widehat{c}$ as we did in class, so that they represent capital per effective worker and consumption per effective worker, respectively. Derive a pair of differential equations in the variables $\left(\widehat{k}, \widehat{c}\right)$.

Notice that the equations in part (c) are exactly the same as those we derived by combining the 9 equations together for the equilibrium version of the model. Therefore, the answer here follows exactly the same steps that we did in class, which leads to

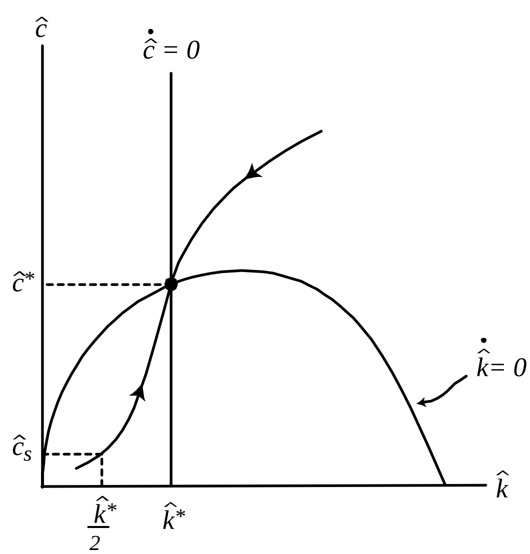
$$\begin{aligned}
\dot{\widehat{c}}(t) &= \frac{1}{\theta}\left[f'\left(\widehat{k}(t)\right) - \delta - \rho - \theta g\right]\widehat{c}(t) \\
\dot{\widehat{k}}(t) &= f\left(\widehat{k}(t)\right) - \widehat{c}(t) - (\delta + n + g)\,\widehat{k}(t)
\end{aligned}$$

d) In class we derived the equations for $\widehat{k}$ and $\widehat{c}$ that characterize the equilibrium of the Ramsey model with technological progress. How does your answer to (c) compare to the equations we derived in class? Why?

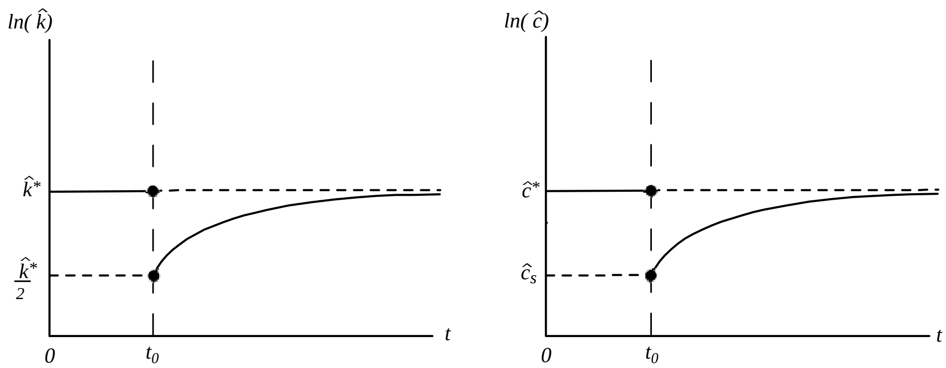
The answers are exactly the same, because the welfare theorems hold and therefore the equilibrium allocation (derived in class) must be equal to the optimal allocation (derived here).

2) Use the model in question (1) to analyze the following situation. Suppose an economy is initially on its balanced growth path, with $\left(\widehat{k}, \widehat{c}\right) = \left(\widehat{k}^*, \widehat{c}^*\right)$. Then, at some point in time $t_0 > 0$, suddenly and unexpectedly, the value of k falls by one-half. (Imagine, for example, that there is a war or a natural disaster and half of the factories are destroyed.) Draw the time paths of k and c from time 0 onwards.

The war causes no change in the phase diagram, other than moving the economy from $\widehat{k}^*$ to $\frac{1}{2}\widehat{k}^*$.



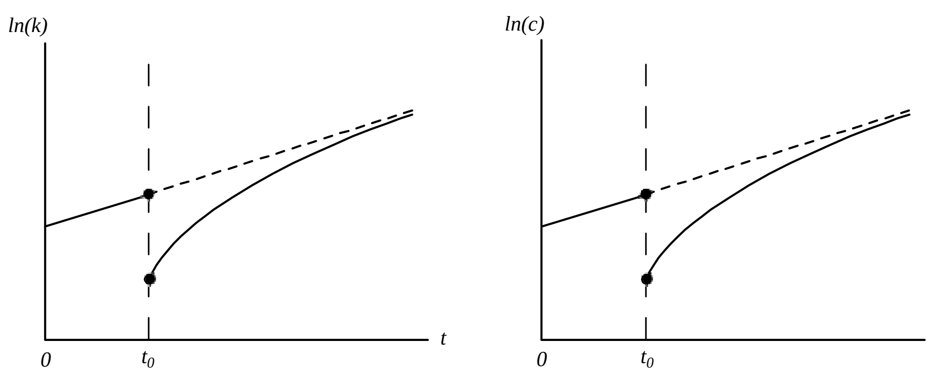
The time paths of $\widehat{k}$ and $\widehat{c}$ are therefore the following:



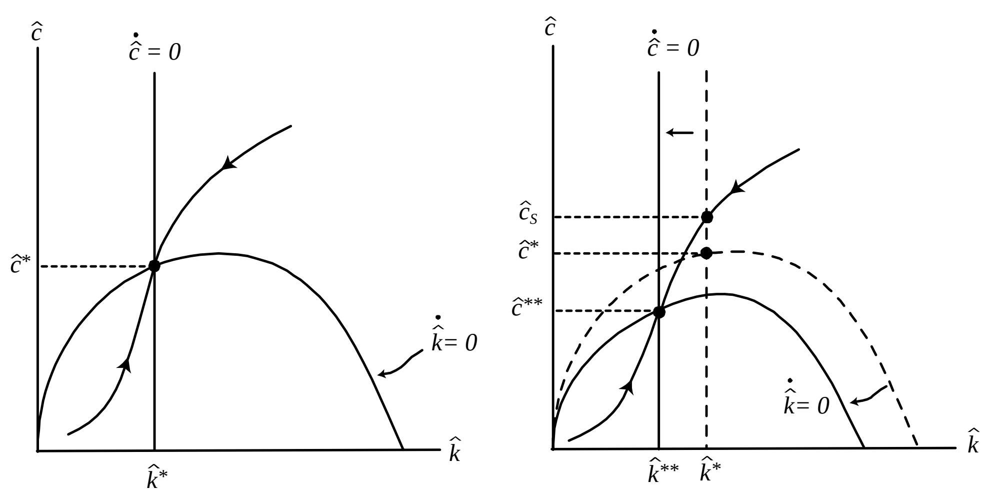
We can translate these graphs into time paths for k and c using the following relationships

$$k = A\widehat{k} \quad \Rightarrow \quad \gamma_k = \gamma_{\widehat{k}} + g \quad \text{and} \quad c = A\widehat{c} \quad \Rightarrow \quad \gamma_c = \gamma_{\widehat{c}} + g.$$

When $\widehat{k}$ and $\widehat{c}$ jump down, so do k and c. The levels of k and c then climb back up to the original balanced growth path.

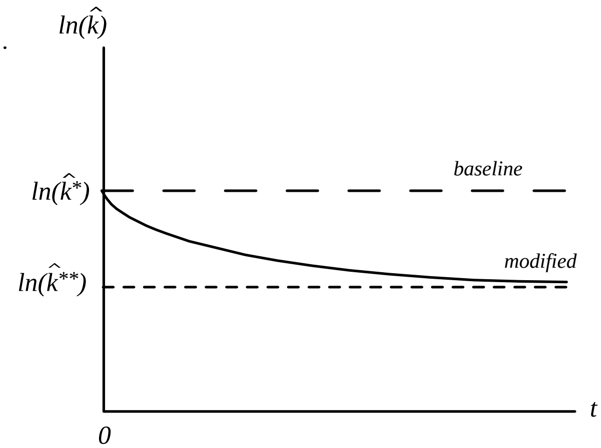


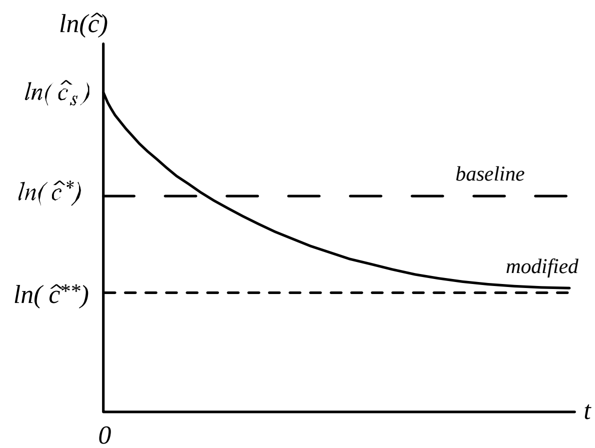
3) Again using the problem in question (1), do the following comparative dynamics exercise: $\delta' > \delta$. Draw the time paths for the variables k and c for the baseline and the modified cases.



A higher value of δ causes the isocline for $\widehat{c}$ to shift to the left, and the isocline for $\widehat{k}$ to rotate downward. These changes are shown in the phase diagram above.

In class, we did the exercise $\delta' < \delta$ using the model without productivity growth. The changes here are essentially the opposite of what we saw in that exercise. The income effect and the substitution effect again point in opposite directions. With an higher value of δ, future consumption is now relatively more expensive compared to current consumption (because the technology for saving is worse), and therefore the substitution effect points toward consuming more today. A higher value of δ also makes the economy poorer, so the income effect points toward consuming less at every point in time, including today. If the substitution effect dominates, $\widehat{c}$ (and therefore c) must initially be higher in the modified case, as in indicated in the phase diagram. Then the time paths of $\widehat{k}$ and $\widehat{c}$ are:





Using the relationships

$$\gamma_k = \gamma_{\hat{k}} + g \quad \text{and} \quad \gamma_c = \gamma_{\widehat{c}} + g'$$

we can draw the time paths of k and c as follows: