Solutions to Problem Set #5

Economic Growth
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Professor Todd Keister
keister@itam.mx

In this problem, you will examine the effect of aid for development when the differences between economies are due to differences in the efficiency of the financial sector. The production function is the usual

$$Y(t) = F\left[K(t), A(t)\, L(t)\right],$$

where the level of productivity $A(t)$ *grows at the constant rate* $g > 0$. *Everything is standard except:*

(i) *One unit of output that is not consumed becomes* $\sigma < 1$ *units of capital. As we have discussed before, we can think of* σ *as measuring the efficiency of the financial sector (that is, how efficiently saving is transformed into productive investment).*

(ii) *At every point in time* t, *the economy receives* $\phi A(t)$ *units of capital as an aid payment.*

a) Write down the complete optimal growth problem.

The optimal growth problem is

$$\begin{aligned}
& \max_{\{c(t)\}} \int_0^\infty \frac{c(t)^{1-\theta} - 1}{1-\theta} e^{-(\rho-n)t} dt \\
& \text{subject to} \\
\dot{k}(t) &= \sigma\left[F(k(t), A(t)) - c(t)\right] - (\delta + n)\, k(t) + \phi A(t), \\
k(0) &= k_0, \text{ and} \\
k(t),\ c(t) &\geq 0 \text{ for all } t.
\end{aligned}$$

b) Write the Hamiltonian function associated with this problem. Use the first-order conditions for this problem to derive a pair of differential equations in the variables c *and* k.

The Hamiltonian function is given by

$$H(t) = \frac{c(t)^{(1-\theta)} - 1}{1-\theta} e^{-(\rho-n)t} + \mu(t)\left[\sigma\left[F(k(t), A(t)) - c(t)\right] - (\delta + n)\, k(t) + \phi A(t)\right].$$

The FOC are given by

$$\begin{aligned}
&\text{(a)} \frac{\partial H}{\partial c} = 0 \quad \Rightarrow \quad c(t)^{-\theta} e^{-(\rho-n)t} = \sigma\mu(t) \\
&\text{(b)} \frac{\partial H}{\partial k} = -\dot{\mu} \quad \Rightarrow \quad \mu(t)\left[\sigma F_K(k(t), A(t)) - (\delta + n)\right] = -\dot{\mu}(t) \\
&\text{(c)} \frac{\partial H}{\partial \mu} = \dot{k} \quad \Rightarrow \quad \dot{k}(t) = \sigma\left[F(k(t), A(t)) - c(t)\right] - (\delta + n)\, k(t) + \phi A(t)
\end{aligned}$$

The transversality condition is

$$\lim_{t\to\infty} \mu(t)k(t) = 0.$$

Following the usual steps, we get

$$\begin{aligned}\dot{c}(t) &= \frac{1}{\theta}\left[\sigma F_K\left(k(t), A\left(t\right)\right) - \delta - \rho\right] c(t) \\ \dot{k}(t) &= \sigma\left[F\left(k\left(t\right), A\left(t\right)\right) - c\left(t\right)\right] - \left(\delta + n\right) k\left(t\right) + \phi A\left(t\right)\end{aligned}$$

c) Define the variables $\hat{k}$ and $\hat{c}$ as in class. Derive the differential equations for these variables.

Again following exactly the same steps that we did in class, we get

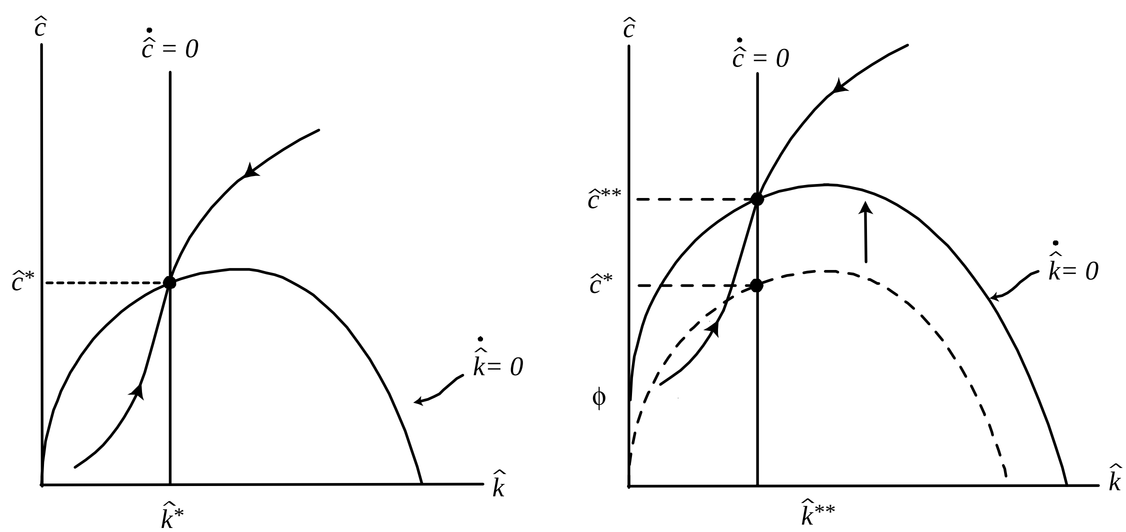
$$\begin{aligned}\dot{\widehat{c}}(t) &= \frac{1}{\theta}\left[\sigma f'\left(\widehat{k}(t)\right) - \delta - \rho - \theta g\right]\widehat{c}(t) \\ \dot{\widehat{k}}(t) &= \sigma\left[f\left(\widehat{k}(t)\right) - \widehat{c}(t)\right] - \left(\delta + n + g\right)\widehat{k}(t) + \phi\end{aligned}$$

Notice that the steady-state level of capital per worker will satisfy

$$\sigma f'\left(\widehat{k}^*\right) = \delta + \rho + \theta g.$$

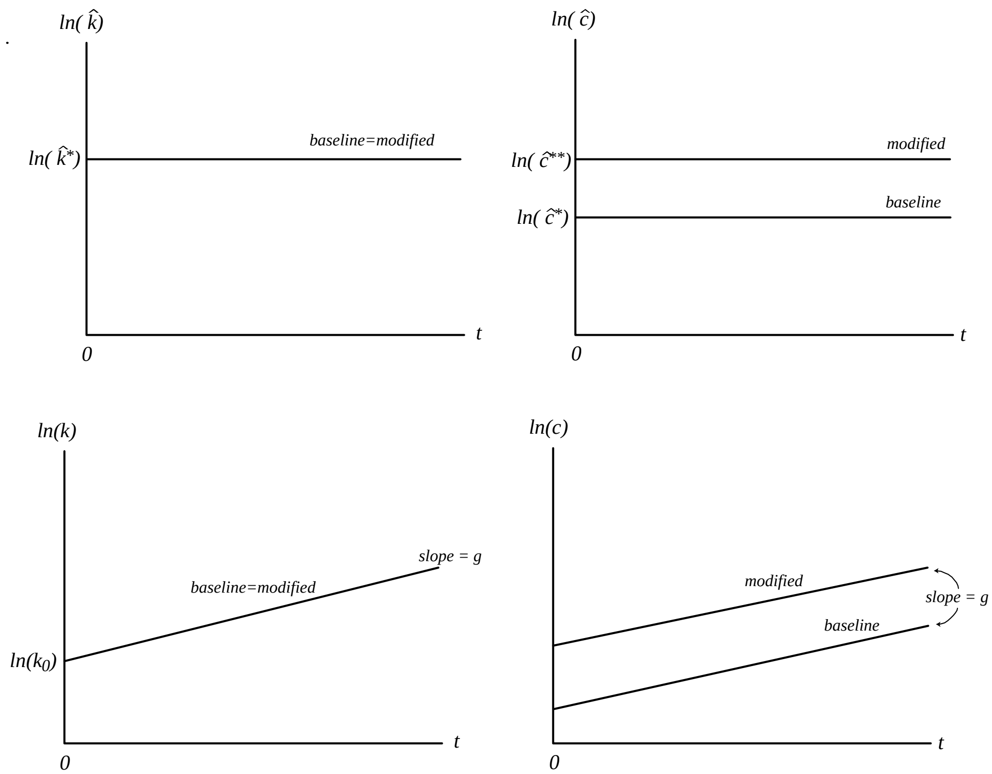
A lower value of σ implies that $f'\left(\widehat{k}^*\right)$ must be higher, and thus that $\widehat{k}^*$ must be lower. This shows that an inefficient financial sector will lead the economy to have a lower steady-state level of capital per effective worker, and hence to be poorer than an economy with a more efficient financial sector.

d) Do the following comparative dynamics exercise: The baseline economy has $\phi = 0$ (no aid) and the modified economy has $\phi > 0$. Draw the phase diagram for both the baseline and the modified economy, indicating what is different. Be sure to label your diagram clearly.

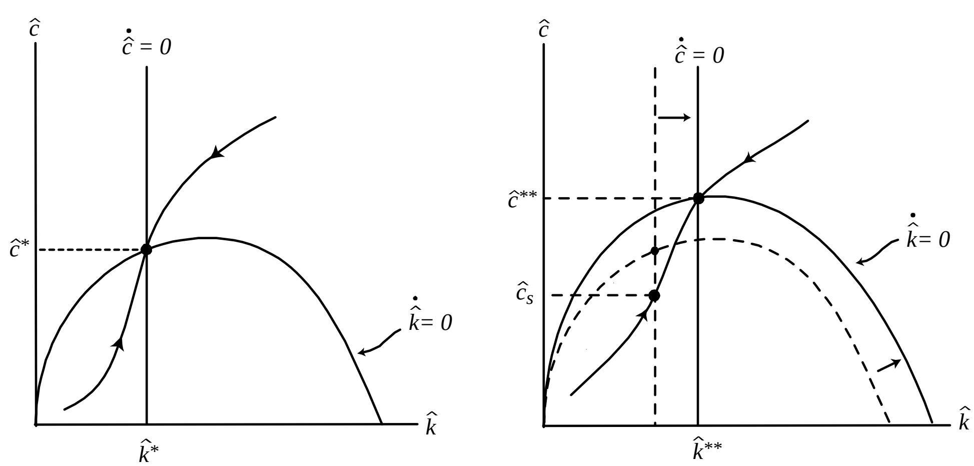


e) Draw the time paths of k and c for both the baseline and the modified economy. Be sure to label the curves and slopes on your diagrams clearly. The diagrams show the predicted effect of aid for development.

I will first draw the time paths of $\widehat{k}$ and $\widehat{c}$, and then the time paths of k and c :



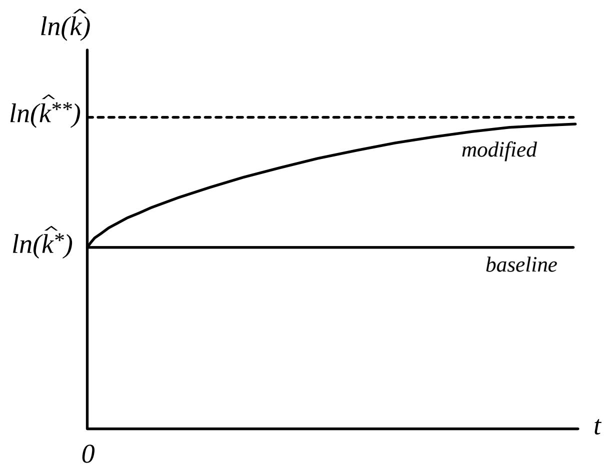
f) Now do a different comparative dynamics exercise (still using your answer to part (c)): $\sigma' > \sigma$. Assume $\phi = 0$ in both cases. Now, the economy is receiving no aid, and we are examining the predicted effect of an increase in the efficiency of the financial sector. Draw the phase diagram for both the baseline and the modified economy, indicating what is different. Be sure to label your diagram clearly. If necessary, assume that the substitution effect is larger than the income effect.

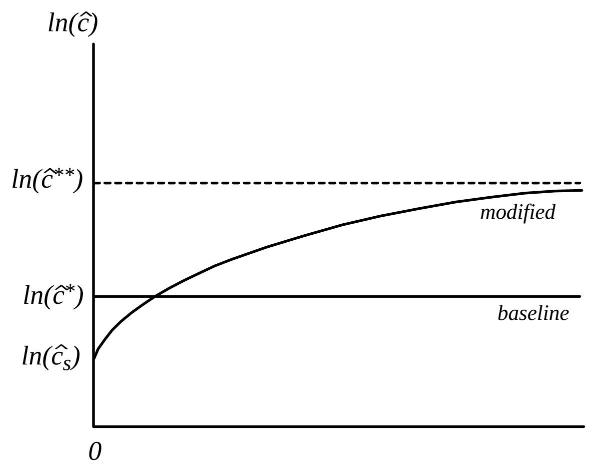


This is similar to the question in Problem Set #2, except now there is productivity growth in the model. When the financial sector becomes more efficient (that is, when σ increases), investing becomes more attractive. Then consuming today is relatively more expensive, so the substitution effect is to consume less today (and more in the future). At the same time, the economy is richer, so the income effect is to consume more at every point in time (including today). If the substitution effect, dominates, we have $\widehat{c}_S < \widehat{c}^*$, as in the diagram above.

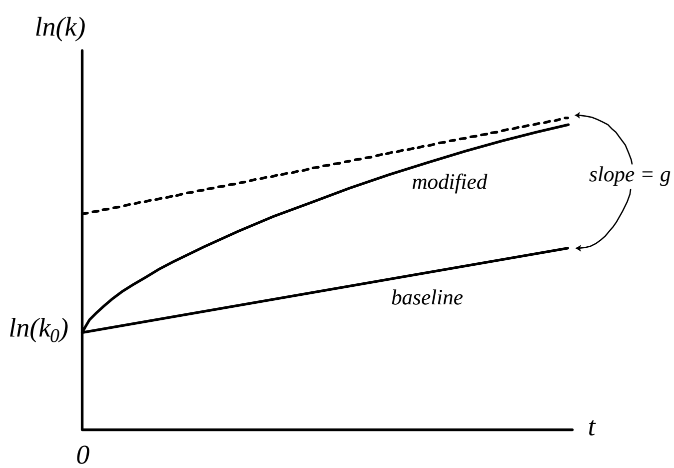
g) Draw the time paths of k and c for both the baseline and the modified economy. Be sure to label the curves and slopes on your diagrams clearly.

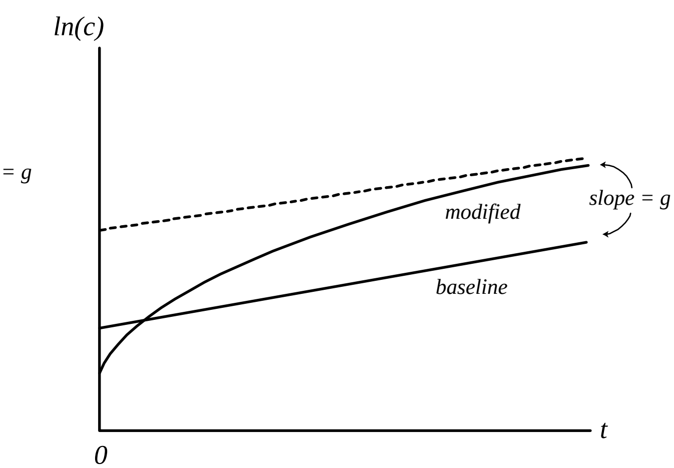
First, the time paths of $\widehat{k}$ and $\widehat{c}$:





Now, the time paths of k and c :





The exercise shows, as we would expect, that improving the efficiency of the financial sector has a positive impact on economic development, while giving aid without improving the financial sector does not.