Solutions to Problem Set #6

Economic Growth
Spring 2005

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Consider the model of human capital. Output is produced according to the production function $Y = K^{\alpha}(huL)^{1-\alpha}$, *where* $0 < \alpha < 1$ *and where* u *is the fraction of time that each person spends working. A constant fraction* s *of output is invested in new physical capital, so that physical capital accumulation is given by*

$$\dot{K}(t) = sK(t)^{\alpha}(h(t)uL(t))^{1-\alpha} - \delta K(t).$$

Human capital accumulation is given by

$$\dot{h}(t) = (1-u)h(t).$$

The labor force $L(t)$ *grows at the constant rate* $n > 0$.

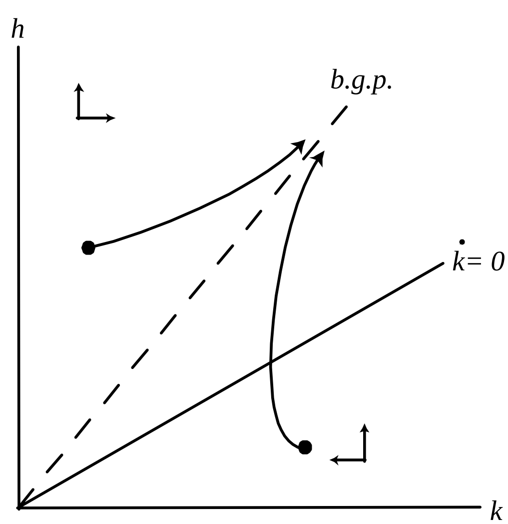
a) Derive the differential equations for $k = \frac{K}{L}$ *and* $\widehat{k} = \frac{k}{h}$.

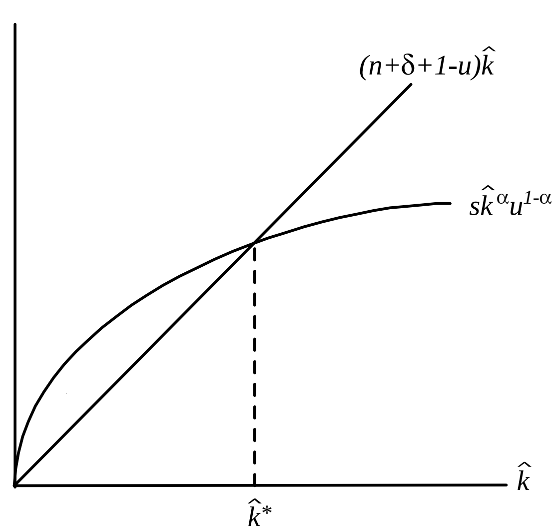
This is exactly as in class:

$$\begin{aligned} \dot{k}(t) &= sk(t)^{\alpha}(h(t)u)^{1-\alpha} - (n+\delta)k(t) \\ \dot{\widehat{k}}(t) &= s\widehat{k}(t)^{\alpha}u^{1-\alpha} - (n+\delta+1-u)\widehat{k}(t) \end{aligned}$$

b) Draw the phase diagram for (k,h) *and the Solow diagram for* $\widehat{k}$. *Be sure to label all of the lines and curves in your graphs.*

This is also as in class.



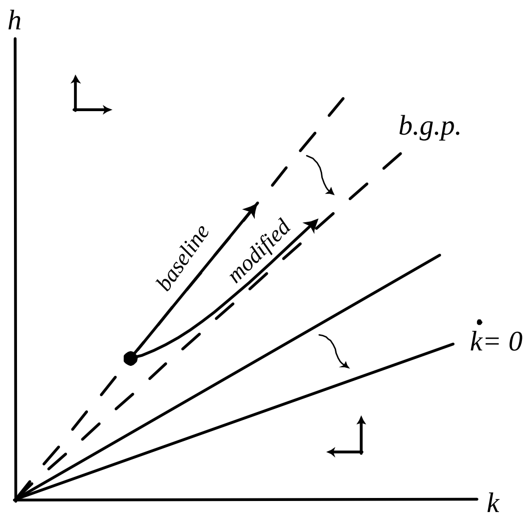


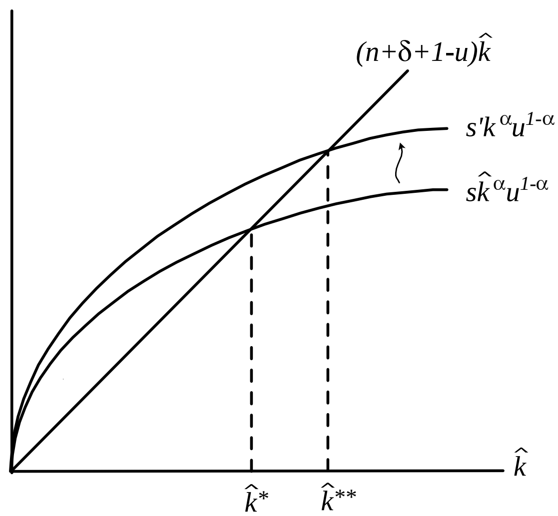
c) We are going to do the following comparative dynamics exercise: $s' > s$.

The exercise takes the usual form. The baseline economy has savings rate s *and is on the balanced growth path at* $t = 0$*. The modified economy starts at* $t = 0$ *with the same amounts of physical capital and human capital as the baseline economy, but with the savings rate* s'*.*

Draw the modified phase diagram for (k, h) *and Solow diagram for* $\widehat{k}$*, indicating what has changed.*

An increase in s leads to a higher steady-state level of $\widehat{k}$. This implies that the slope of the balanced growth path becomes lower. The isocline for k also becomes flatter, as shown in the picture below.





d) Draw the time paths of (the logs of) h*,* k*, and* y *for both the baseline and the modified economy. Pay particular attention to the slopes of these functions right at* $t = 0$.

The time path of h does not depend on s, and hence is the same for both the baseline and the new case. Next, using the relationship

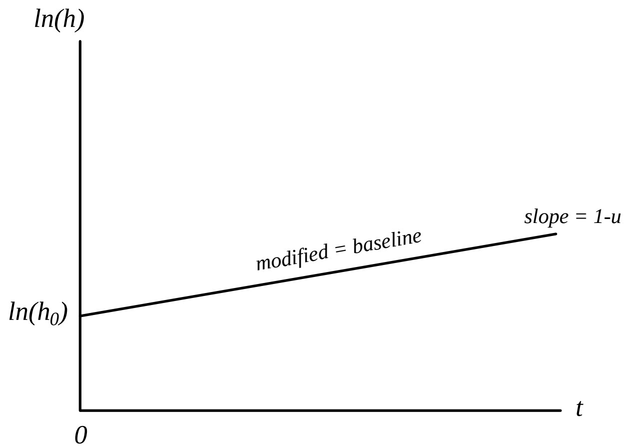
$$\gamma_k = \gamma_{\widehat{k}} + \gamma_h$$

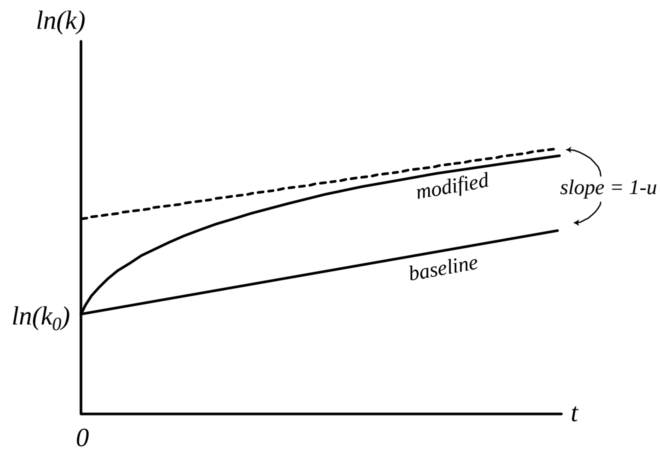
and the above diagram for $\widehat{k}$, we can draw the time path for k. Initially k is growing at a rate greater than $1 - u$, as $\widehat{k}$ grows up to the new steady-state level $\widehat{k}^{**}$. In the long run, however, the growth rate of k must return to $1 - u$.

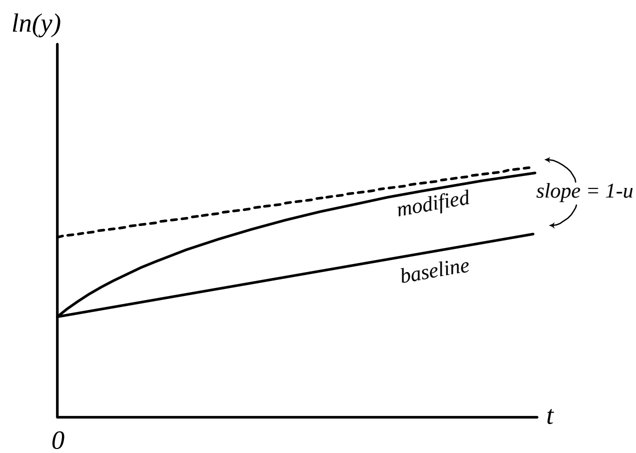
For the time path of y, we use

$$\gamma_y = \alpha\gamma_k + (1 - \alpha)\,\gamma_h.$$

In other words, the growth rate of output is a weighted average of the growth rates of the inputs. Initially, therefore, y is also growing at rate faster than $1 - u$, but in the long run the growth rate returns to $1 - u$.







Suppose we did the exercise $s' > s$ in the Solow model with exogenous productivity growth. What would the time paths of k and y look like? You might want to work through this exercise; the answer is that they will look exactly the same as the time paths drawn above. Why? Our model of human capital is in many ways very similar to the Solow model. The primary difference is that the growth rate of skills in the human capital model is affected by the variable u, which measures the division of time between working and learning. (In the Solow model, in contrast, the growth rate of productivity is completely exogenous to the model.) In exercises like the one above where u is not changed, therefore, our model of human capital will behave very similarly to the Solow model.