Solutions to Problem Set #7

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Consider the Ak model, where there are externalities in production. As in class, the production function of the representative firm is given by

$$Y = AK^{\alpha}L^{1-\alpha}\overline{K}^{1-\alpha},$$

where $\overline{K}$ is the total amount of capital in the economy. Consumers have the usual utility function

$$u(c) = \frac{c^{(1-\theta)} - 1}{1-\theta}.$$

Assume there is no population growth $(n = 0)$, and normalize the population to $N = 1$.

We saw in class that the government could make the equilibrium optimal by providing a subsidy on capital rental by firms. Suppose that the government tries a different policy: it subsidizes savings by households. *In particular, for each unit of assets that the household owns at time t, the government gives the household a payment of σ (this is in addition to the payment $r(t)$ that the household receives from the bank). Assume that σ is constant through time. To finance this expenditure, the government taxes labor income at time t at rate $\tau(t)$.*

a) Write the household's maximization problem and derive the differential equations for the variables c and a.

$$\max \int_0^{\infty} \frac{c(t)^{(1-\theta)} - 1}{1-\theta} e^{-\rho t} dt$$

subject to

$$\dot{a}(t) = (1-\tau(t))\, w(t) + (r(t) + \sigma)\, a(t) - c(t),$$
$$a(0) = a_0,$$
$$a(t) \geq -B, \quad \text{and} \quad c(t) \geq 0 \quad \text{for all } t.$$

Setting up the Hamiltonian function and taking FOC is straightforward. The resulting differential equations are

$$\dot{c}(t) = \frac{1}{\theta}\left[r(t) + \sigma - \rho\right] c(t) \tag{1}$$
$$\dot{a}(t) = (1-\tau(t))\, w(t) + (r(t) + \sigma)\, a(t) - c(t). \tag{2}$$

The first equation reflects the fact that the return to saving is now equal to the market interest rate plus the subsidy, while the second equation is simply a restatement of the budget constraint.

b) Write the maximization problem of a typical firm and solve this problem to obtain the rental rate and wage as functions of k and $\bar{k}$.

Firms are not directly affected by this policy, so their problem is standard

$$\max AK(t)^{\alpha}L(t)^{1-\alpha}\overline{K}(t)^{1-\alpha} - w(t)L(t) - R(t)K(t),$$

as are the first-order conditions

$$\begin{aligned} \alpha Ak(t)^{\alpha-1}\overline{K}(t)^{1-\alpha} &= R(t) && (5)\\ (1-\alpha)\,Ak(t)^{\alpha}\overline{K}(t)^{1-\alpha} &= w(t). && (6) \end{aligned}$$

c) What are the equilibrium conditions for this economy?

$$\begin{aligned} L(t) &= N(t) = 1 && (7)\\ a(t) &= k(t) && (8)\\ r(t) &= R(t) - \delta && (9)\\ \overline{K}(t) &= K(t) = k(t) && (10) \end{aligned}$$

d) Assume the government has a balanced budget at each point in time. What is the government's budget constraint?

$$\tau(t)w(t) = \sigma a(t) \qquad \text{or} \qquad \tau(t) = \frac{a(t)}{w(t)}\sigma. \tag{11}$$

e) Use the information from the previous parts to derive equilibrium differential equations for the variables c and k. (Note: these equations will depend on the level of subsidy σ).

Performing the usual (messy) substitutions leads to

$$\begin{aligned} \dot{c}(t) &= \frac{1}{\theta}\left[\alpha A - \delta + \sigma - \rho\right] c(t)\\ \dot{k}(t) &= (A-\delta)\,k(t) - c(t). \end{aligned}$$

f) *Can the level of subsidy σ be chosen so that the equilibrium is optimal? If so, what value of σ does this? What is the implied tax rate on labor income $\tau(t)$?*

Yes. If $\sigma = (1-\alpha)\,A$, then the two differential equations will be the same as the ones resulting from the optimal growth problem (and hence the equilibrium will be optimal). Using the government budget constraint, we can see that the tax rate $\tau(t)$ on labor will then be

$$\begin{aligned} \tau(t) &= \frac{k(t)}{w(t)}\sigma(t)\\ &= \frac{1}{(1-\alpha)\,A}\,(1-\alpha)\,A\\ &= 1. \end{aligned}$$

This implies that the after-tax wage will be zero when the optimal policy is implemented.