Solutions to Problem Set #8

Economic Growth — Professor Todd Keister
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Consider the Romer model, where output is produced using different types of capital according to the production function

$$Y(t) = AL(t)^{1-\alpha} \int_0^{M(t)} K_j(t)^{\alpha} \, dj.$$

Households have the usual preferences and there is no population growth (normalize $N = 1$). Suppose the government subsidizes the sale *of capital: for each unit of K_j that firm j sells, the government pays it an amount σ (hence, the total revenue that firm j gets from selling one unit of output is $R_j + \sigma$). The government finances these subsidies by taxing labor income.*

a) What are the differential equations for the variables c and a that come from solving the household's problem under this policy?

The household's problem is standard, so we have

$$\begin{aligned} \dot{c}(t) &= \frac{1}{\theta}\left[r(t) - \rho\right] c(t) && (1) \\ \dot{a}(t) &= (1 - \tau(t))\, w(t) + r(t)\, a(t) - c(t) && (2) \end{aligned}$$

b) Write the profit-maximization problem of a final output producer, and use this problem to derive the demand function for capital of type j.

$$\max \; AL^{1-\alpha} \int_0^M K_j^{\alpha} dj - wL - \int_0^M R_j K_j dj$$

The first-order condition with respect to K_j is

$$\alpha A L^{1-\alpha} K_j^{1-\alpha} = R_j. \qquad (5)$$

This is the demand function for capital of type j.

c) Write the profit-maximization problem of firm j, and use this problem to derive the production intensity K^* (the amount of each type of capital that will be used in production).

These firms receive the subsidy. For each unit of capital they produce, the receive a payment of R_j in the market and an additional payment of σ from the government. Their problem, therefore is

$$\max \; (R_j + \sigma)\, K_j - K_j$$

or

$$\max \; \alpha A L^{1-\alpha} K_j^{\alpha} - (1 - \sigma)\, K_j.$$

The first-order condition is

$$\alpha^2 A L^{1-\alpha} K_j^{\alpha-1} = 1 - \sigma,$$

which can be solved for

$$K^* = \left(\frac{\alpha^2 A}{1-\sigma}\right)^{\frac{1}{1-\alpha}} L.$$

The implied price is given by

$$R_j^* = \frac{1-\sigma}{\alpha}.$$

This leads to profits at time t of

$$\pi(t) = \left(\frac{1-\alpha}{\alpha}\right)\left(\alpha^2 A\right)^{\frac{1}{1-\alpha}} (1-\sigma)^{\frac{\alpha}{\alpha-1}} L.$$

d) What are the equilibrium conditions? Use these conditions to determine the equilibrium interest rate for this economy.

$$L(t) = N(t) = 1 \tag{7}$$
$$a(t) = \eta M(t) \tag{8}$$
$$\eta = \frac{1}{r(t)}\left(\frac{1-\alpha}{\alpha}\right)\left(\alpha^2 A\right)^{\frac{1}{1-\alpha}} (1-\sigma)^{\frac{\alpha}{\alpha-1}} L \tag{9}$$

The interest rate is then given by

$$r(t) = \frac{1}{\eta}\left(\frac{1-\alpha}{\alpha}\right)\left(\alpha^2 A\right)^{\frac{1}{1-\alpha}} (1-\sigma)^{\frac{\alpha}{\alpha-1}}$$

e) What is the equilibrium growth rate of consumption for this economy? (This will depend on the levels of subsidy σ.)

$$\gamma_c = \frac{1}{\theta}\left[\frac{1}{\eta}\left(\frac{1-\alpha}{\alpha}\right)\left(\alpha^2 A\right)^{\frac{1}{1-\alpha}} (1-\sigma)^{\frac{\alpha}{\alpha-1}} - \rho\right]$$

f) Can the government choose the levels of subsidy σ so that the equilibrium is optimal? If so, what is the correct level? If not, why not?

No. The level of σ that generates the correct production intensity K^* is $\sigma = (1-\alpha)$. However, this generates the growth rate

$$\gamma_c = \frac{1}{\theta}\left[(\alpha)\frac{1}{\eta}\left(\frac{1-\alpha}{\alpha}\right)(\alpha A)^{\frac{1}{1-\alpha}} - \rho\right],$$

which is lower than the optimal growth rate (because of the first α). The reason for this is that, under this policy, the monopolists earn too little profit (yes, subsidizing the monopolist directly gives her less profit than subsidizing her customers as we did in class). Because of this, fewer people engage in research, the demand for funds is low, the interest rate is low, and therefore the growth rate is lower than the optimal level.