Technical Appendix

The following is a supplemental appendix to our paper "Understanding Monetary Policy Implementation."[1] This appendix contains a more formal derivation of the demand curves presented in the figures in the main text. We formulate the profit function of a typical bank under each of the different policy regimes and derive the bank's optimal choice of reserve position. We also derive some properties of the resulting demand curve for reserves in each case.

A.1 The benchmark case

We begin with the benchmark case, which corresponds to Figure 1 in the text. Recall that, in this case, no interest is paid on reserve balances and there are no fees for daylight credit. If the bank's final reserve balance falls below the requirement $K,$ the difference must be borrowed at the penalty rate $r_P.$ Since r is the market interest rate, a bank's opportunity cost of holding a quantity R of reserve balances is given by the product $rR.$ The change in a typical bank's profits associated with its reserve operations can, therefore, be written as

$$\pi = -rR - \int_{R-K}^{\infty} r_P \left(P - (R - K)\right) f\left(P\right) dP,$$

where f is the density function for the late-day payment shock. Notice all of the terms in this expression are negative; when no interest is paid on reserve balances, reserve operations can only serve to lower a bank's profit. The bank is willing to incur these costs because it is required to hold reserves and make payments as a part of its (generally profitable) operations.

The bank will choose its reserve holdings R to maximize the value of π. The first-order condition for this problem is

$$\frac{\partial \pi}{\partial R} = -r + r_P \int_{R-K}^{\infty} f\left(P\right) dP = 0,$$

which can be solved for

$$r = r_P \left(1 - F\left(R - K\right)\right).$$

In other words, the optimal level of reserve balances equates the opportunity cost of holding one more unit of reserves with the marginal change in expected reserve deficiency costs. This latter change comes not from having a deficiency less often (which does happen, but is not a first-order effect), but rather from having a smaller deficiency when the payment shock P is high. The marginal change is, therefore, equal to the penalty rate r_P multiplied by the probability of a deficiency $\left(1 - F\left(R - K\right)\right).$ To put things slightly differently, the height of the demand curve in figure 1 is, for any given value of $R,$ equal to the marginal change in expected deficiency costs evaluated at $R.$

The slope of the demand curve in Figure 1 is given by

$$\frac{\partial r}{\partial R} = -r_P f\left(R - K\right). \tag{1}$$

This expression shows that the slope of the demand curve for reserves is proportional to the height

[1] Huberto M. Ennis and Todd Keister, "Understanding Monetary Policy Implementation," Federal Reserve Bank of Richmond *Economic Quarterly* Vol. 94, No. 3 (Summer 2008), pp 235-263.

of the density function for the payment shock. When the distribution of the shock is uniform, the slope of the demand curve is thus constant, as depicted in Figure 1. Under different distributional assumptions, the demand curve may have more "curvature", but the overall shape will remain similar. In particular, for any distribution with support $\left[-\overline{P}, \overline{P}\right]$, the demand curve will be flat at r_P until the point $K - \overline{P}$ and will be flat on the horizontal axis after the point $K + \overline{P}$. Between these two points, the demand curve will always be downward sloping. Different distributions merely change the shape of this downward-sloping part of the curve.[2]

Suppose, for example, that the distribution of the late-day payment shock is hump-shaped, like the solid curve in the left panel of Figure 8. In this case, moderate values of P are more likely to occur than extreme values near either $-\overline{P}$ or $\overline{P}$. Using equation (1), it is easy to see that the corresponding demand curve must look like that depicted in the right panel of the figure, with a small slope for values near $K - \overline{P}$ and $K + \overline{P}$, but a steeper slope around the point K. Intuitively, because the probability of a payment shock near $-\overline{P}$ is very small, the bank is less concerned about a large payment inflow that would leave it holding excess reserves at the end of the day. As a result, the bank is willing to hold a larger quantity of reserves when the interest rate is high, which is why the demand curve in the right-hand panel lies above the dashed line for values of the overnight rate near r_P. The bank is also less concerned about a large payment outflow – that is, a realization near $\overline{P}$ – that might leave its end-of-day balance below K. It will choose, therefore, to hold fewer reserves than in the uniform case when the interest rate is near zero.

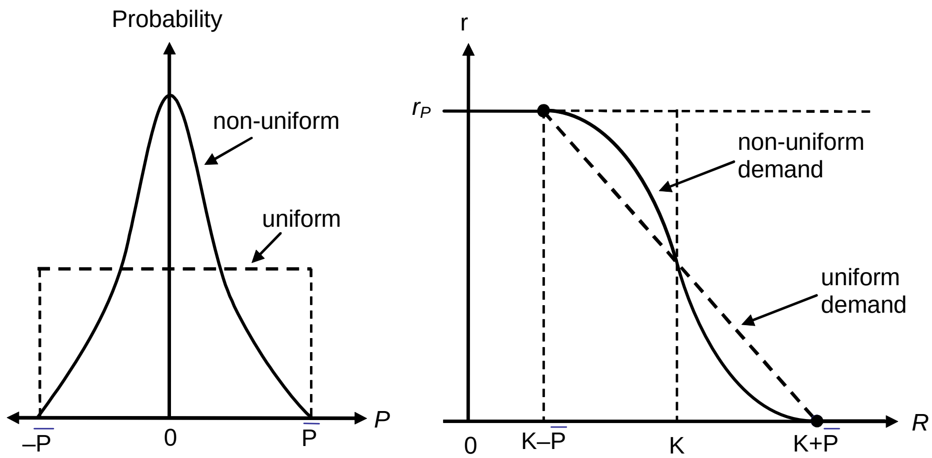

Figure 8: Slope of the demand curve

A.2 Interest rate corridors

Now suppose that the central bank remunerates reserve balances at a rate $r_D > 0$. In this case, a

[2] If the shock instead had an unbounded distribution, such as the normal distribution used by Whitesell (2006) and others, the demand curve would again have this same shape, but would asymptote to the rate r_P and to the horizontal axis without ever intersecting them.

bank's profits associated with its reserve operations can be written as

$$\begin{aligned}\pi &= -rR + \int_{-\infty}^{R-K} r_D \left((R-K) - P \right) f(P)\, dP \\ &\quad - \int_{R-K}^{\infty} r_P \left(P - (R-K) \right) f(P)\, dP + r_D K.\end{aligned} \tag{2}$$

The final term in this expression indicates that the bank must hold enough reserve balances to meet its requirement K, and that it will earn interest at rate r_D on these balances. If, after the payment shock is realized, the bank is holding excess reserves, those will also be compensated at rate r_D; these situations are captured in the first integral in the equation. The second integral captures the situations where the shock is larger than $R-K$ and the bank must borrow at the penalty rate r_P to meet the requirement. Notice that reserves borrowed from the discount window and used to meet requirements are remunerated at the rate r_D and thus have a net cost of $(r_P - r_D)$.

The optimal reserve position of the bank is characterized by the first-order condition

$$\frac{\partial \pi}{\partial R} = -r + r_D \int_{-\infty}^{R-K} f(P)\, dP + r_P \int_{R-K}^{\infty} f(P)\, dP = 0.$$

The optimal choice now equates the opportunity cost of holding one more unit of reserves, r, with the marginal change in expected reserve deficiency costs *plus* the marginal change in expected interest income . Solving for the demand curve yields

$$r = r_D + (r_P - r_D)\left(1 - F(R-K)\right). \tag{3}$$

Here we see that the demand curve will never fall below the interest rate paid on reserves r_D, as depicted in Figure 6. The slope of the demand curve is given by

$$\frac{\partial r}{\partial R} = -\left(r_P - r_D\right) f(R-K).$$

As in the benchmark case, we see that this slope is proportional to the height of the density function for the payments shock.

It is interesting to note that the interest rate paid on required reserves has no effect on the demand curve. This can be seen from the profit function (2), where the interest revenue from required reserves appears as a fixed, additively-separable payment. In the model studied here, where the reserve requirement is fixed independently of a bank's actions, remunerating reserves at a below-market rate simply acts as a lump-sum tax on banks and has no effect on bank behavior.

A.3 Reserve maintenance periods

We now examine the case of a two-day maintenance period, as studied in Section 5 above. We assume that excess reserves are remunerated at rate r_D and, for simplicity, that reserve balances held to meet requirements are not remunerated.[3] Let π_1 denote the net profit earned by the bank on the first day of the maintenance period, and let R_1 denote the bank's choice of reserve position on

[3] As discussed above, the remuneration rate on required reserves has no effect on the demand curves in our model.

that day. Then we have

$$\pi_1 = -r_1 R_1 + \int_{-\infty}^{R_1 - 2K} r_D \left(R_1 - 2K - P\right) f\left(P\right) dP - \int_{R_1}^{\infty} r_P \left(P - R_1\right) f\left(P\right) dP, \tag{4}$$

where r_1 denotes the market interest rate on first day. If the bank experiences a large late-pay payment inflow $\left(P < R_1 - 2K\right)$, it will satisfy its entire requirement for the period on the first day. In this case, any reserves held beyond the required amount are remunerated at rate r_D. If the bank experiences a large late-day payment outflow $\left(P > R_1\right)$, the bank will be forced to borrow at the penalty rate in order to avoid having an overnight overdraft. For intermediate values of the payments shock, however, the bank will neither have a deficit nor accumulate any excess reserves; its reserve balance at the end of the day is simply applied toward the total requirement.

Let R_2 denote the bank's reserve holdings on the second (and final) day of the maintenance period and r_2 the market interest rate on that day. Let π denote the total expected profit at the end of the maintenance period. Then we can write

$$\begin{aligned} \pi &= \pi_1 + r_2\left(\pi_1 - R_2\right) + \int_{-\infty}^{R_2 - K_2} r_D \left(R_2 - K_2 - P\right) f\left(P\right) dP \\ &\quad - \int_{R_2 - K_2}^{\infty} r_P \left(P - \left(R_2 - K_2\right)\right) f\left(P\right) dP, \end{aligned} \tag{5}$$

where

$$K_2 = \left\{ \begin{array}{c} 2K \\ 2K - \left(R_1 - P_1\right) \\ 0 \end{array} \right\} \text{ if } R_1 - P_1 \left\{ \begin{array}{c} \leq 0 \\ \in \left(0, 2K\right) \\ \geq 2K \end{array} \right\}$$

and P_1 denotes the realization of the bank's payment shock on the first day. The variable K_2 measures the remaining requirement to be met on that day (if any), which typically equals the total requirement $2K$ minus the bank's end-of-day balance on the first day $(R_1 - P_1)$. Following the steps in the previous subsection, the demand curve on the last day of the maintenance period is easily seen to be

$$r_2 = r_D + \left(r_P - r_D\right)\left(1 - F\left(R_2 - K_2\right)\right).$$

Notice that is expression depends on first-day variables (r_1 and R_1) only through their effect on K_2. Also note that the bank's optimal choice of R_2 will move one-for-one with the remaining requirement K_2, that is, $dR_2/dK_2 = 1$ holds in the relevant region.

On the first day of the maintenance period, the bank will choose R_1 in order to maximize expected profits, given its belief about the interest rate that will prevail on the second day. Assume, for simplicity, that the bank has perfect foresight about the rate r_2. We have already shown that the choice of R_1 does not affect the difference $\left(R_2 - K_2\right)$. Therefore, this choice has no effect on the last two terms in the expression for total profit (5). In effect, then, the bank's reserve position on the first day is chosen to solve

$$\max_{R_1} \; \pi_1 + r_2\left(\pi_1 - E\left[R_2\left(R_1; P_1\right)\right]\right),$$

where R_2 will be chosen optimally given K_2, which depends on R_1 and the realization of P_1. In other words, the bank chooses the quantity of reserves it holds on the first day to maximize its profit on the first day, taking into account the effect this choice will have on its reserve holdings on the second day. Using the solution for the second day derived above, we can show the relationship between R_1 and R_2 to be characterized by

$$\frac{dR_2}{dR_1} = \frac{dR_2}{dK_2}\frac{dK_2}{dR_1} = \left\{ \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right\} \text{ for } R_1 \left\{ \begin{array}{c} < P_1 \\ \in (P_1, P_1 + 2K) \\ > P_1 + 2K \end{array} \right\}.$$

Using this relationship and substituting in for π_1 from (4) yields

$$\max_{R_1} \; (1 + r_2) \left(-r_1 R_1 + \int_{-\infty}^{R_1 - 2K} r_D \left(R_1 - 2K - P\right) f\left(P\right) dP - \int_{R_1}^{\infty} r_P \left(P - R_1\right) f\left(P\right) dP \right)$$
$$-r_2 \left(\int_{-\infty}^{R_1 - 2K} \underline{R}_2 \left(P\right) dP + \int_{R_1 - 2K}^{R_1} R_2 \left(R_1; P\right) f\left(P\right) dP + \int_{R_1}^{\infty} \overline{R}_2 f\left(P\right) dP \right),$$

where $\underline{R}_2$ is the quantity of reserves the bank will choose to hold on the second day if $K_2 = 0$ and $\overline{R}_2$ is the corresponding quantity for $K_2 = 2K$. Both of these numbers are constants, independent of the choice of R_1.

The first-order condition for this problem can be written as

$$-r_1 + r_D \int_{-\infty}^{R_1 - 2K} f\left(P\right) dP - r_P \int_{R_1}^{\infty} f\left(P\right) dP - \frac{r_2}{1 + r_2} \int_{R_1 - 2K}^{R_1} f\left(P\right) dP \frac{dR_2}{dR_1} = 0.$$

The first part of this expression is similar to the earlier first-order conditions: it reflects the opportunity cost of holding reserves, r_1, as well as the marginal changes in expected deficiency costs and expected interest earnings on the first day. The last term in the expression is new; it reflects the expected effect of first-day reserve holdings on second-day reserve holdings. This condition can be solved for the demand function

$$r_1 = r_P - \left(r_P - \frac{r_2}{1 + r_2} \right) F\left(R_1\right) - \left(\frac{r_2}{1 + r_2} - r_D \right) F\left(R_1 - 2K\right). \tag{6}$$

This function corresponds to the demand curve depicted in Figure 5.

To see why the demand curve in (6) generates the shape presented in Figure 5, first consider very low (i.e., negative) values of R_1. If R_1 is small enough, both $F\left(R_1\right)$ and $F\left(R_1 - 2K\right)$ will be zero (or very close to zero). From (6), the corresponding market interest rate would then be r_P. In other words, the demand curve is initially flat at the level r_P, as depicted in the figure. Next consider the other extreme case, where R_1 is large enough that both $F\left(R_1\right)$ and $F\left(R_1 - 2K\right)$ are close to unity. In this case, the corresponding market interest rate is equal to r_D; hence, the demand curve is eventually flat at level r_D, again as depicted in the figure. Finally, suppose that K is large enough so that for some intermediate values of R_1, we have both

$$F\left(R_1\right) \approx 1 \quad \text{and} \quad F\left(R_1 - 2K\right) \approx 0.$$

For these values of $R_{1,}$ the demand curve lies at

$$\frac{r_2}{1+r_2} \approx r_2.$$

Note the approximation here. In deriving Figure 5, we said that a bank would be indifferent between holding reserves on the two days if $r_1 = r_2$ holds. This is not quite correct, since the bank should discount the opportunity cost of holding reserves on the second day. However, for reasonable values of the daily interest rate, this discounting is immaterial. (Formally, r_2 is the best first-order approximation of $r_2/(1+r_2)$ around the point $r_2 = 0$.) Hence, for intermediate values of R_1, the demand curve will be flat at a value very close to r_2 as long as the total requirement $2K$ is large enough. In such cases, the demand curve in (6) looks precisely like the one depicted in Figure 5.

A.4 Clearing Bands

Now suppose that a bank has a single-day reserve requirement with a clearing band of the type discussed in Section 6.2. The bank must hold a minimum reserve balance $\underline{K}$ at the end of the day, borrowing at the penalty rate if necessary to make up any deficiency. The bank will earn the target rate of interest r_T on all balances up to some limit $\overline{K} > \underline{K}$. Above $\overline{K}$, all reserves are remunerated at a lower rate r_D, which could be zero. In other words, the bank will earn the target rate of interest r_T on all of its reserves as long at the total falls in the clearing band $\left[\underline{K}, \overline{K}\right]$. Outside of this clearing band, the costs and benefits are set as in a channel system.[4]

A bank's expected profit associated with its reserve operations under this system is

$$\begin{aligned}\pi &= -rR + \int_{-\infty}^{R-\overline{K}} \left(r_T\overline{K} + r_D\left(R-P-\overline{K}\right)\right) f(P)\,dP \\ &\quad + \int_{R-\overline{K}}^{R-\underline{K}} r_T(R-P)\,f(P)\,dP + \int_{R-\underline{K}}^{\infty} \left(r_T\underline{K} - r_P\left(P-\left(R-\underline{K}\right)\right)\right) f(P)\,dP.\end{aligned}$$

The first integral in this expression captures situations where the late-pay payment shock is small enough that the bank's final reserve balance is greater than $\overline{K}$ (this might, for example, happen if the bank experiences a large late-day payment inflow). In such instances, the bank earns the rate r_T and the first $\overline{K}$ reserves and the rate r_D on the remainder. The second integral captures intermediate values of the payment shock, which leave the bank's final reserve balance between $\overline{K}$ and $\underline{K}$, in which case the bank earns the rate r_T on all of these balances. The third integral captures large payment outflows that leave the bank's final reserve balance below $\underline{K}$. In these cases, the bank must borrow at the penalty rate r_P to meet the minimal requirement $\underline{K}$.

As before, the bank will choose R in order to maximize expected profit. The first-order condition for this problem can be written as

$$\frac{\partial\pi}{\partial R} = -r + r_D\int_{-\infty}^{R-\overline{K}} f(P)\,dP + r_T\int_{R-\overline{K}}^{R-\underline{K}} f(P)\,dP + r_P\int_{R-\underline{K}}^{\infty} f(P)\,dP = 0.$$

Once again, the optimal choice of reserve position involves balancing the opportunity cost of hold-

[4] Note that if $\underline{K} = \overline{K}$, this system becomes a channel system with rate r_T paid on reserves held to meet requirements.

ing reserves, r, against the marginal changes in both expected deficiency costs and expected interest receipts. Solving for the demand curve yields

$$r = r_D + (r_T - r_D)\left(1 - F\left(R - \overline{K}\right)\right) + (r_P - r_T)\left(1 - F\left(R - \underline{K}\right)\right). \tag{7}$$

This demand curve corresponds to the one presented in Figure 7. Its slope is given by

$$\frac{\partial r}{\partial R} = -\left(r_T - r_D\right) f\left(R - \overline{K}\right) - \left(r_P - r_T\right) f\left(R - \underline{K}\right).$$

The understand the shape of this curve, first consider values of R that are low enough that both $F\left(R - \overline{K}\right)$ and $F\left(R - \underline{K}\right)$ are zero (or very close to zero). In such cases, the interest rate emerging from (7) is the penalty rate r_P. In other words, the demand curve is initially flat at r_P. Next, consider very large values of R, so that both $F\left(R - \overline{K}\right)$ and $F\left(R - \underline{K}\right)$ are equal to unity. In these cases, the interest rate from (7) is r_D, meaning that the demand curve is eventually flat at this level. Finally, consider intermediate values of R. If the clearing band $\left[\underline{K}, \overline{K}\right]$ is wide enough, there will exist some values of R such that

$$F\left(R - \overline{K}\right) \approx 0 \quad \text{and} \quad F\left(R - \underline{K}\right) \approx 1.$$

For these values, (7) shows that the demand curve will be flat at the target rate r_T, as depicted in Figure 7.